THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering

Final Examination (2021/2022)
MHZ3551: Engineering Mathematics I

Date: 31st January 2023 (Tuesday) **Time: 0930 hrs. – 1230 hrs.**

Instructions:

- *Answer **five (05)** questions only.*
- *Number of pages in the paper is **four (04)**.*
- *All the symbols are in standard notation unless they are defined.*

Q1. (a). Use the method of contradiction to prove the following statement. Let $x, y \in \mathbb{R}$ (30%)

$$\forall x \forall y \left(7x + 15y > 15 \Rightarrow \left(x > \frac{8}{7} \vee y > \frac{7}{15} \right) \right).$$

(b). Prove that for any $a \in \mathbb{Z}$, $9a^2 + 3a + 14$ is divisible by 2 by using the proof by case. (40%)

(c). Use the method of contrapositive to prove the following statement. (30%)

Let $x, y, c \in \mathbb{R}$ and $a, b \in \mathbb{R}^+$.

$$\forall x \forall y \left(ax + by > c \Rightarrow \left(x > \frac{c}{a+b} \vee y > \frac{c}{a+b} \right) \right).$$

Q2. (a). Let A and B be any two subsets of the universal set U. Using the laws of sets prove that $A - (A - B) = A \cap B$. (20%)

(b). Show that the relation defined on $\mathbb{Z}^+$ such that $\sigma = \{ (x, y) \mid y = nx, n \in \mathbb{Z}^+ \}$ is a partially order relation. (20%)

(c). If x is an independent variable, then show that the relation $x^2 - 4x - 3 = 0$ is a function. Is this function a one-to-one function? Justify your answer. (30%)

(d). If $x \mapsto f(x) = x^2$ for $x \in (0, 2)$ and $x \mapsto g(x) = 2x - 3$ for $x \in (0, 2)$ then find the range of f, R_f and the range of g, R_g. Find the function $f \circ g$. (30%)

Q3. (a). Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -2 & 1 \end{bmatrix}$.

(i). Find the matrices AB and BA. (20%)

(ii). Consider the following system of linear equations.

$$x + 2y + z = 1$$
$$2x + 3y + z = 4$$
$$3x + 4y + 2z = 4.$$

(α). Prove that $AX = C$, where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 4 \\ 4 \end{bmatrix}$. Using part (i), find the values of x, y and z. (30%)

(β). Hence, deduce the solutions of the following system of linear equations. (20%)

$$x + 2y + z = 5$$
$$2x + 3y + z = 10$$
$$3x + 4y + 2z = 13.$$

(b). Let $A = \begin{bmatrix} -1 & a \\ b & 1 - ab \end{bmatrix}$ and $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ where $a, b \in \mathbb{R}$

(i). Find the characteristic equation of A. Hence, prove that $A^2 = I_2 - abA$. (20%)

(ii). Deduce that $A^{-1} = A + abI_2$. (10%)

Q4. (a). $(a_n)_{n\in Z^+}$ be a sequence of positive numbers such that $a_1 > \sqrt{7}$ and $a_{n+1}(7 + a_n) = 7(1 + a_n)$ for each $n \geq 2$.

(i). Express $a_{n+1}^2 - 7$ in terms of a_n. Use the principle of mathematical induction to prove that $a_n > \sqrt{7}$ for each $n \geq 1$. (25%)

(ii). Deduce that $(a_n)_{n\in Z^+}$ is a strictly decreasing sequence and find $\lim_{n\to\infty} a_n$. (15%)

(b). Use D' Alembert's ratio test to show that $\sum_{n=1}^{\infty} \frac{2022^n + 2021}{2020^n}$ is divergent. (20%)

(c). Show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2 + y^4}$ does not exist. (20%)

(d). Show that $\frac{1}{\lambda^2}\frac{\partial^2 f}{\partial x^2} + \frac{1}{\mu^2}\frac{\partial^2 f}{\partial y^2} = 0$ if $f(x,y) = e^{\lambda x}\sin(\theta + \mu y)$. where θ is a constant. (20%)

Q5. (a). Let f be a function such that $f(x) = \tan^{-1}\left(\frac{x}{k}\right)$, where $k \in \mathbb{R}\backslash\{0\}$.

(i). By using the mean value theorem prove that there exits $c \in (a, b)$ such that (30%)

$$\frac{k(b-a)}{c^2+k^2} = \tan^{-1}\left(\frac{b}{k}\right) - \tan^{-1}\left(\frac{a}{k}\right) \text{ for any } (a, b) \subset \mathbb{R}$$

(ii). Hence prove that for any $(a, b) \subset \mathbb{R}^+$ and $k \in \mathbb{R}^+$ (20%)

$$\frac{k(b-a)}{b^2+k^2} < \tan^{-1}\left(\frac{b}{k}\right) - \tan^{-1}\left(\frac{a}{k}\right) < \frac{k(b-a)}{a^2+k^2}.$$

(iii). Hence deduce that (10%)

$$\frac{\pi}{4} + \frac{1}{5} < \tan^{-1}(2) < \frac{\pi}{4} + \frac{1}{2}.$$

(b). Let f be a function such that $f(x) = (x^2 - 17ax + 70a^2)e^{-\frac{x}{2a}}$, where $a \in \mathbb{R}^+$. (40%)
Prove that

$$\frac{df(x)}{dx} = -\frac{1}{2a}(x - 13a)(x - 8a)e^{-\frac{x}{2a}}.$$

Verify Roller's theorem.

Q6. (a). Find the general solution of the following differential equation. (30%)

$$\frac{dy}{dx} = -\frac{6x + 7y}{9y + 7x}.$$

(b). Find the general solution of the following differential equation. (30%)

$$\frac{dy}{dx} + 3y = 8e^{-3x}\cos 9x.$$

(c). Show that the differential equation $y(1 + xy)dx + x(1 - xy)dy = 0$ is not exact. (40%)
If it has an integrating factor of the form $x^n y^m$ find the values of m, n. Hence, find the general solution of the differential equation.

Q7. (a). Let $y_p = ke^x$ be a particular solution of the differential equation,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 2021e^x.$$

(i). Find the value of k. (20%)

(ii). Obtain the general solution of the above differential equation. (15%)

(b). Consider the differential equation $\frac{d^2y}{dx^2} + 20\frac{dy}{dx} + 96y = 560e^{2x}$.

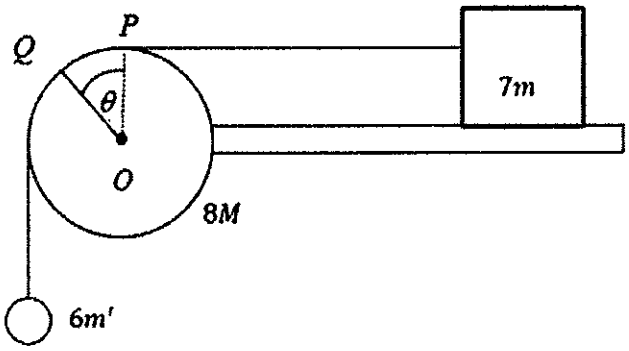
(i). By using D −operater methods, find a particular integral. (25%)

(ii). Find the general solution of the above differential equation. (15%)

(iii). If $\left(\frac{dy}{dx}\right)_{x=0} = -28$ when $x = 0$ and $y = 10$, then find the solution of the differential equation. (25%)

Q8. (a). A pulley of mass 8M and radius a is fixed to the edge of a smooth horizontal table. The pulley can rotate freely in a vertical plane about a horizontal axis through its center. (50%)

A light inextensible string runs over the pulley, connecting a block of mass 7m, to a block of mass 6m′. The block of mass 7m is held at rest on the table with block of mass 6m′ hanging freely, as shown in the diagram. The system is released from rest while the strings are taught. In the subsequent motion, the block of mass 7m moves on the table. When the



pulley rotates an angle θ it has an angular acceleration $\ddot{\theta}$. Assume that the string does not slip, that does not reach the pulley and that the moment of inertia of a uniform circular of mass m and radius a is $\frac{1}{2}ma^2$. Find the tensions of the strings and the angular acceleration.

(b). A spherical ice-ball falls under gravity through still air. As it falls, it acquires moisture from the air, causing it to increase in mass.
Assume that the ball remains spherical throughout the motion and no external force other than the gravitational force acting on the ball.
Let at time t seconds, the radius of the ball is r, its mass is m, and the radius is increasing at the rate of λr, where λ is a positive constant. Let the density of the ball is ρ where a positive constant.

(i). Show that at time $t, \frac{dm}{dt} = 4\pi\lambda\rho r^3$. (10%)

(ii). If the velocity of the ball is v at time t, then show that (15%)

$$v\frac{dm}{dt} + m\frac{dv}{dt} = mg.$$

(iii). If $v = 0$ when $t = 0$, then show that (25%)

$$v = \frac{g}{3\lambda} + \left(u - \frac{g}{3\lambda}\right)e^{-3\lambda t}.$$

Deduce the terminal velocity of the ball.

End

-Copyright reserved -