The Open University of Sri Lanka
B.Sc. / B. Ed Degree Programme
Final Examination – 2023/2024
Pure Mathematics – Level 05
Number Theory – PEU5303

Date: 26.03.2024 Time:01.30 p.m. – 03.30 p.m.

Answer **Four** Questions only.

(01)

(a) State the well ordering principle.

(i). Let $A = \{x + y + z + v:\ x, y, z, v, w \in \mathbb{N},\ x^2 + y^2 + z^2 + v^2 = w^3\}$. Prove that the set A has a least element.

(b) Prove the following:

(i). If $d|n$ and $n|m$, then $d|m$,

(ii). If d and n are positive and $d|n$, then $d \leq n$,

(iii). If the product of two integers is even, then at least one of them must be even.

(c) Using the Principle of Mathematical Induction, verify the identity $6|n(2n + 1)(n + 1)$ for each $n \in \mathbb{N}$.

(02)

(a) State, without proof, the division algorithm.

Using the division algorithm prove that $6|n(n + 1)(n + 2)$ for each $n \in \mathbb{Z}$.

(b) Use Euclidean Algorithm to find $\gcd(24, 138)$. Hence find $x, y \in \mathbb{Z}$ such that $\gcd(24, 138) = 24x + 138y$.

(c) Construct a linear Diophantine equation to solve following word problem.

Find all integer solutions to that Diophantine equation. Hence find all positive solutions to the word problem.

A customer bought a dozen (12) pieces of fruit, apples, and oranges, for 132 rupees. If an apple costs 3 rupees more than an orange and there are more apples than oranges, how many pieces of each kind were bought?

(03)

(a) (i). Prove that every integer greater than 1 is prime or can be expressed as a product of primes.

(ii). Determine whether the integer 701 is prime, by using the Sieve of Eratosthenes.

(b) Prove the following:

(i). If $a \equiv b(\text{mod } n)$, then $a^k \equiv b^k(\text{mod } n)$ for any positive integer k,

(ii). $111^{333} + 333^{111}$ is divisible 7.

(c) Find all incongruent solutions to the following Linear congruences.

(i). $9x \equiv 21(\text{mod } 30)$,

(ii). $18x \equiv 30(\text{mod } 42)$.

(04)

(a) The school marching band has more than 100 members, but fewer than 300 members. When they line up in rows of 4 there is one extra student, when they line up in rows of 5 there are two extra students and when they line up in rows of 7 there are three extra students. Use Chinese remainder to find how many students are in the band.

(b) State the Fermat's Little Theorem.

(i). Use the above theorem to verify that 17 divides $11^{104} + 1$.

(ii). Let p and q are distinct primes with $a^p \equiv a(\text{mod } q)$ and $a^q \equiv a(\text{mod } p)$. Prove that $a^{pq} \equiv a(\text{mod } pq)$.

(iii). Show that $2^{31} \equiv 2(\text{mod } 11)$ and $2^{11} \equiv 2(\text{mod } 31)$. Deduce that $2^{341} \equiv 2(\text{mod } 341)$.

(05)

(a) Show that the set $\{-12, -4, 11, 13, 22, 82, 91\}$ is a complete set of residues modulo 7.

(b) Is the set $\{-31, -16, -8, 13, 25, 80\}$ form a reduced set of residues modulo 9? Justify your answer.

(c) State and prove Wilson's Theorem.

(i). Let p is a prime. Prove that for any integer a,

$p|a^p + (p-1)!\,a$ and $p|(p-1)!\,a^p + a$.

(ii). Find the remainder when $4\,(29)! + 5!$ is divided by 31.

(06)

(a) Solve the Diophantine equation $43x + 5y = 250$ by using continued fraction.

(b) Prove that every finite simple continued fraction represents a rational number.

(c) Define the Euler phi function $\varphi(n)$.

(i). Let p be a prime and k be a positive integer. Prove that

$\varphi(p^k) = p^k\left(1 - \frac{1}{p}\right)$.

(ii). Find $\varphi(n)$ for $n = 2730$ and 4080.

(iii). Find the last two digits of decimal expansion of 3^{256}.

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