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The Open University of Sri Lanka
B.Sc./B.Ed. Degree Programme
Final Examination - 2024/2025
Pure Mathematics – Level 05

**PEU5303 – Number Theory**

**Duration: Two hours.**

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**Date: 29-04-2025** **Time: 9:30 a.m. - 11:30 a.m.**

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Answer **four** (04) Questions **only**.

**Question 01**

(a) Let $S_0 = d$ and $S_n = 3S_{n-1} + c$, for $n \geq 1$, where $c$ and $d$ are real constants.

(i) Calculate the first 5 terms of $S_n$.
(ii) By Observing these terms, guess a closed form formula for the $n^{th}$ term.
(iii) Using the Principle of Mathematical Induction, prove your guess.

(b) Let $a, b \in \mathbb{Z}$ and $a, b \neq 0$. Prove that if $a|b$ and $b|a$ then $a = b$ or $a = -b$.

**Question 02**

(a)

(i) State, without proof, the Division Algorithm.
(ii) Prove that $4 \nmid (m^2 + 2)$ for any $m \in \mathbb{Z}$.

(b)

(i) Using the Euclidean Algorithm find the greatest common divisor of 25 and 15. hence find $x, y \in \mathbb{Z}$ such that $25x + 15y = \gcd(25,15)$.

(ii) Amali has two types of stamps, stamps worth of Rs.25 and stamps worth of Rs.15. By assuming she has plenty of stamps in both types, find all possible combinations to make collection of stamps worth of Rs.500.



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**Question 03**

(a) Prove that if $a \equiv b(mod\ n_1)$ and $a \equiv c(mod\ n_2)$, then $b \equiv c(mod\ n)$, where $n = \gcd(n_1, n_2)$.

(b) Find all incongruent solutions for each of the following linear congruences:
(i) $5x \equiv 8(mod\ 14)$.
(ii) $7x \equiv 2(mod\ 126)$.

(c) A school hosting a sport event and students need to be grouped for different activities. When students are divided into,
groups of 5 , then 25 students were left out,
groups of 8 , then 05 students were left out and
groups of 11, then 08 students were left out.
Using the Chinese Remainder Theorem, find the smallest possible number of students participated in this event.

**Question 04**

(a) Let $P(x) = \sum_{k=0}^{m} c_k x^k$ be a polynomial of $x$ where $c_k \in \mathbb{Z}$, for every $k \in \{0, 1, 2, \ldots, m\}$. If $a \equiv b(mod\ n)$, prove that $P(a) \equiv P(b)(mod\ n)$.

(b) Let $M = a_0 + 10a_1 + 10^2a_2 + 10^3a_3 + 10^4a_4 + 10^5a_5$ where $0 \leq a_i \leq 9$ for $i = 0, 1, 2, 3, 4, 5$ ; be the decimal representation of positive integer $M$.
Let $T = a_0 + 10a_1 + a_2 + 10a_3 + a_4 + 10a_5$. Show that $M \equiv T(mod\ 99)$.

(c)
(i) State, without proof, the Fermat's Little Theorem.
(ii) Use the above theorem to verify that 17 divides $3^{164} + 4$.

(d)
(i) State without proof the Wilson's theorem.
(ii) Find the remainder when 17! is divided by 19.

**Question 05**

(a) Define a complete set of residues modulo n.

(b)

(i) For any integer $k$, prove that $\{-k+0, -k+1, -k+2, -k+3, \ldots, -k+n-1\}$ forms a complete set of residues modulo $n$.

(ii) Deduce that $\{\frac{-m+1}{2}, \frac{-m+3}{2}, \ldots\ldots, \frac{m-3}{2}, \frac{m-1}{2}\}$ forms a complete set of residues modulo $m$, where $m$ is an odd integer.

(c) Verify that $\{0,1,2,2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9\}$ form a complete set of residues modulo 11 but that $\{0,1,2,2^2, 2^3, 2^4, 2^5, 2^6, 2^7\}$ does not form a complete set of residues modulo 9.

**Question 06**

(a) Define $\tau(n)$, where $n$ is a positive integer.

(b) (i) Let $n \in \mathbb{N}$. If $n$ is a perfect square, prove that $\tau(n)$ is odd.
[Hint: If $n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \ldots . p_k^{e_k}$ is the prime factorization of $n > 1$, then $\tau(n) = (e_1 + 1)(e_2 + 1)(e_3 + 1) \ldots (e_n + 1)$ ].

(ii) Find the value of $\tau(144)$.

(c) (i) Define $\sigma(n)$ where $n$ is a positive integer.
(ii) Let $p$ be a prime number and $k$ be a positive integer. Prove that $\sigma(p^k) = \frac{p^{k+1}-1}{p-1}$.

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