THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering

Bachelor of Technology Honors in Engineering

Final Examination (2016/2017)
MPZ5134: The Nature of Mathematics

Date: 14st November 2017 (Tuesday) **Time: 1330 hrs. – 1630 hrs.**

Instructions

- ***Answer five (05) questions only.***
- ***Number of pages in the paper is four (04).***
- ***State clearly any assumptions you required.***
- ***All the symbols are in standard notation unless they are defined.***

Some important notes:

- Axioms for real numbers

For arbitrary real numbers a, b, c:

A0.	$a + b$ and $a.b$ real numbers	M1.	$ab = ba$
A1.	$a + b = b + a$	M2.	$a(bc) = (ab)c$
A2.	$a + (b + c) = (a + b) + c$	M3.	$a.1 = a$
A3.	$a + 0 = a$	M4.	$a.\frac{1}{a} = 1$ if $a \neq 0$
A4.	$a + (-a) = 0$	D.	$a(b + c) = ab + ac$

- Order Axioms

  P1 – If a, b in $\mathbb{P}$, then $a + b$ in $\mathbb{P}$

  P2 – If a, b in $\mathbb{P}$, then ab in $\mathbb{P}$

  P3 – For any real numbers a, b of $\mathbb{R}$, one and only one of the following is true:

  a in $\mathbb{P}$, $a = 0$ $-a$ in $\mathbb{P}$.

- We write $a < b$ for $b - a$ in $\mathbb{P}$.
- Integer a divides b $a|b$ if and only if $b = ca$ for some integer c.

- Mathematical Induction:

  If we can prove both of the following steps, we then know that the proposition is a result (Theorem), true for all n.

(I) The initial step:

Prove the proposition is true for $n = 1$

(II) Inductive step:

Assume the proposition is true for $n = k$, and on this basis, prove it is true for $n = k + 1$.

---------- **End of the notes** ----------

Questions:

Q1. (a). Let $a, b, c \in \mathbb{R}$. Using the axioms of real numbers and the order axioms, prove that: (40%)

(i). $-(-a) = a$.

(ii). $a + b < a + c$ if and only if $b < c$.

(iii). $-(a + b) = -a - b$.

(b). (i). Using (a) (i), prove that if $-b < -a$, then $a < b$. (20%)

(ii). Using (a) (i) and (b) (i), prove that if $a < b$, then $-b < -a$.

(c). Solve the absolute value inequality $0 < |2x - 5| < 3$ for all the possible solutions, and draw the related graph. (25%)

(d). Let n be a positive integer. Using the Mathematical Induction, prove that (15%)

$$1^3 + 2^3 + 3^3 + \cdots + (n-1)^3 + n^3 = \frac{n^2(n+1)^2}{4} \text{ for all } n.$$

Q2. (a). Using the truth tables, prove the following equivalences: (25%)

(i). $p \vee p \equiv p$.

(ii). $p \vee q \equiv q \vee p$.

(iii). $p \Rightarrow q \equiv (q \vee \sim p)$.

(iv). $(p \vee q) \vee r \equiv p \vee (q \vee r)$.

(b). (i). Using part (a), deduce that $(p \Rightarrow q) \vee (p \Rightarrow r) \equiv p \Rightarrow (q \vee r)$. (40%)

(ii). Show that $[p \wedge (p \Rightarrow q)] \Rightarrow q$ is a tautology.

(c). Consider the following collection of statements about the integer n: "If n is prime, then n is odd", "n is not prime or n is not odd", "n is odd and n is not prime". (35%)

(i). Express the above collection of statements in the symbolic form with the following proportional notations.

p: n is prime. q: n is odd.

(ii). Determine whether the above collection of statements is consistent or inconsistent.

Q3. (a). By using the method of direct proof, prove the following. (35%)

(i). If $a + b$ is an even integer and b is an odd integer, then ab is an odd integer.

(ii). For $m, n \in \mathbb{Z}$, $m < n$, if and only if $m \leq n - 1$.

(b). By using the method of proof by contradiction, show that: (40%)

(i). Let a and b be integers. If $a^3 + b^3$ is odd, then a is odd or b is odd.

(ii). There is no real solution for the equation $x^2 - 4x + 5 = 0$.

(c). Show that: (25%)

(i). $\forall x \in \mathbb{R}, x^2 + 1 \geq 2x$ is true.

(ii). $\forall x \in \mathbb{R}, |\sqrt{2}x| > |x|$ is false.

Q4. (a). Suppose that A, B and C are sets. Show by the method of Venn diagram that: (60%)

(i). $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$.

(ii). $(A \cap B^c) \cup (A^c \cap B) = (A \cup B) \cap (A \cap B)^c$.

(b). Provide an example to show each of the following: (40%)

(i). If $A \subseteq B$, then $A \cup C \nsubseteq B \cap C$.

(ii). If $A \subseteq (B \cup C)$, then $C \cap B^c \nsubseteq A \cap B^c$.

Q5. (a). Suppose that E is a universal set. Now consider the following nine results for any subset A, B, C of E. (65%)

$R0$: $\emptyset \subseteq A$

$R1$: $A \cap A^c = \emptyset$

$R2$: $A \cap B = B \cap A$

$R3$: $A \cap B \subseteq A$

$R4$: $A = B \Leftrightarrow A \subseteq B \text{ and } B \subseteq A$

$R5$: $A \subseteq B$ and $C \subseteq D \Rightarrow A \cap C \subseteq B \cap D$

$R6$: $(A^c)^c = A$

$R7$: $(A \cup B)^c = A^c \cap B^c$

$R8$: $(A \cap B)^c = A^c \cup B^c$

$R9$: $A \setminus B = A \cap B^c$

$R10$: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Now, using only the above eleven results, the properties of "=", and the algebraic method, prove the following, where A, B, C are arbitrary subsets of E.

(i). $A \setminus B \subseteq B^c$

(ii). $(A \setminus B) \cap (B \setminus A) = \emptyset$

(iii). $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$

(b). By using the logic method, prove the results $R0, R1$ and $R6$ in the above. (35%)

Q6. (a). Let $A = \{1, 2\}$, B = {a, b, c}. (40%)

Define a relation ρ from A to B such that $\rho = \{(1, b), (2, c)\}$ and a relation σ from B to A such that $\sigma = \{(a, 1), (b, 1), (b, 2)\}$.

(i). Find each of the following:

$(\alpha).\ \rho^{-1}$ $(\beta).\ \sigma^{-1}$ $(\gamma).\ \sigma \circ \rho$ $(\delta).\ \rho^{-1} \circ \sigma^{-1}$

(ii). Is $(\sigma \circ \rho)^{-1} = \rho^{-1} \circ \sigma^{-1}$? Justify your answer.

(b). Let R be a relation defined on the set of integers $\mathbb{Z}$ such that xRy, if and only if $xy \geq 0$ for all $x, y \in \mathbb{Z}$. (60%)

(i). Is R an equivalence relation on $\mathbb{Z}$? Justify your answer.

(ii). If R is an equivalence relation on $\mathbb{Z}$, then find the equivalence classes $[5]$ and $[-5]$.

Q7. (a). (i). State the division Algorithm for integers a, b with $b > 0$. (25%)

(ii). Appling the division Algorithm, find *the* $g.c.d(84, 45)$.

(b). (i). State a necessary and sufficient condition for non-zero integers a and b to be relatively prime. (40%)

(ii). Let a, b and c be non – zero integers. Show that:

(α) If $g.c.d(a, b) = d$, then $g.c.d\left(\frac{a}{d}, \frac{b}{d}\right) = 1$

(β) If $b|(c - a)$, then $g.c.d(a, b) = g.c.d(b, c)$.

(c). (i). State the well-ordering principle. (35%)

(ii). Using the well-ordering principle of $\mathbb{N}$. show that

$$\sum_{r=1}^{n} \frac{1}{(2r-1)(2r+1)} = \frac{n}{(2n+1)} \text{ for all } n \in \mathbb{N}.$$

End

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