**The Open University of Sri Lanka**

**B.Sc./B. Ed Degree Programme**

**Final Examination – 2024/2025**

**Pure Mathematics – Level 05**

**PEU5304 – Introduction to Complex Analysis**

**Duration: Two hours.**

**Date: 22.11.2024** **Time:02.00 p.m. – 04.00 p.m.**

Answer **Four** Questions only.

(01) (a) Define the complex number.

Find two complex numbers $z_1, z_2$ such that $\mathrm{Re}(z_1 z_2) \neq \mathrm{Re}(z_1)\,\mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1 z_2) \neq \mathrm{Im}(z_1)\,\mathrm{Im}(z_2)$.

(b) Let $z_1$ and $z_2$ be two complex numbers.

Prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

Deduce that $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$.

(c) Let $z_1 = -12 + 5i$ and $z_2 = 4 - 3i$. Prove that

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$,

(ii) $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\mathrm{Re}(z_1 \bar{z}_2)$.

(02) (a) Let $z_1$ and $z_2$ be two non-zero complex numbers. Prove each of the following:

(i) $\frac{z_1}{z_2}$ is a non-zero complex number.

(ii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$.

(b) Express $\frac{-1+\sqrt{3}i}{5-5i}$ in the polar form.

(c) (i) Let $z_1 = z_2 = \sqrt{3}i$. Verify that $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$.

(ii) Is it true that $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) = \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$ for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$? Justify your answer.

(03) (a) Let $z_1 = 1 + i$ and $z_2 = 1 + \sqrt{3}i$. Construct the points $P_1$ and $P_2$ in the Argand diagram that represent the complex numbers $z_1$ and $z_2$ respectively. Use the points $P_1$ and $P_2$ to construct the point $P$ that represent the complex number $\frac{z_1}{z_1}$.

(b) Find and represent the loci of the points in the complex plane that represent the complex numbers $z$ such that:

(i) $\mathrm{Re}[(1+i)(z-1+i)] = 4$.

(ii) $\mathrm{Im}[z + 2 - 3i] = 5$.

(iii) $|(1+i)z - 2| = 6$.

(04) (a) State the De Moivre's Theorem. Prove the theorem for all positive integers.

Deduce that, the result is true for any negative integer.

(b) Express $\left(\sqrt{5} + 1 + i\left(\sqrt{10 - 2\sqrt{5}}\right)\right)$ in the form of $\cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$.

Hence find the value of $\left(\sqrt{5} + 1 + i\left(\sqrt{10 - 2\sqrt{5}}\right)\right)^{-5}$.

(c) Write down the $n$ distinct roots of $[r_0(\cos\theta_0 + i\sin\theta_0)]^{1/n}$, where $n$ is a positive integer and $r_0 > 0$.

**Hence**, evaluate $[(1+i)^2]^{1/3}$.

(05) (a) Verify that $1 - 2i$ is a root of the polynomial $p(z) = z^4 + 3z^2 + 6z + 10$.

Find all the roots of the equation $p(z) = 0$.

(b) Prove or disprove each of the following statements. Justify your answers.

(i) Let $z_1$ and $z_2$ be two complex numbers. Then, $e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$.

(ii) $e^z$ is real if and only if $\mathrm{Im}z = n\pi, n \in \mathbb{Z}$.

(iii) Let $z_1$ and $z_2$ be two complex numbers. Then,

$\sin z_1 + \sin z_2 = 2\sin\left(\frac{z_1+z_2}{2}\right)\cos\left(\frac{z_1-z_2}{2}\right)$.

(iv) For each $z \in \mathbb{C}$, $\sinh^2 z - \cosh^2 z = 1$.

(06) (a) Define the complex logarithmic number $\log z$.

Find all the roots of the equation $\log z = \left(\frac{17\pi}{2}\right) i$.

(b) Prove or disprove each of the following statements. Justify your answers.

(i) Let $z_1, z_2$ be arbitrary non-zero complex numbers. Then,
$\log(z_1 z_2) \neq \log z_1 + \log z_2$.

(ii) Let for each $n_1, n_2 \in \mathbb{Z}$ if $n_1 \neq n_2$. Then, $e^{\sqrt{2}\left(\frac{\pi}{2}+2n_1\pi\right)i} \neq e^{\sqrt{2}\left(\frac{\pi}{2}+2n_2\pi\right)i}$.

(iii) Let $z_1, z_2$ be non-zero complex numbers and let $c$ be a complex number. Then,
$\left(\frac{z_1}{z_2}\right)^c = \frac{{z_1}^c}{{z_2}^c}$.

(c) Show that for each $z \in \mathbb{C}$, $\sin^{-1} z = -i \log\left[iz + (1 - z^2)^{\frac{1}{2}}\right]$. Evaluate, $\sin^{-1}(2i)$.

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