00143

The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme – Level 04

Final Examination– 2024/2025

Pure Mathematics

PEU4303 – Group Theory 1

Duration: - Two Hours.

Date: - 30.04.2025 Time: - 01.30 p.m. - 03.30 p.m.

Answer Four Questions Only.

(01) (a) Let G be the set of all translation mappings $T_a = \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ given by $T_a(x, y) = (x + a, y + a)$ where $a \in \mathbb{R}$. Show that G forms an infinite abelian group under the composition of mapping.

(b) Let G' be the set of all 2×2 matrices $\begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}$ such that $a \in \mathbb{R}\backslash\{0\}$. Show that G' together with matrix multiplication is a group.

(02) (a) Prove that the set $\{I, (1\,2), (3\,4), (1\,2)(3\,4)\}$ is a subgroup of S_4.

(b) Let G be an abelian group. Let $H = \{x \in G : x = x^{-1}\}$ be the subset of G. Prove that H is a subgroup of G.

(c) If $o(a) = 8$, define the orders of elements a^2, a^3, a^4, a^6 and a^{-1}.

(03) (a) Show that $(S, \otimes)$, where S is the subset $\{1,3,5,7\}$ of $\mathbb{Z}_8^{\times}$ is not cyclic.

(b) Prove that the group $(\mathbb{Z}_{11}^{\times}, \otimes)$ is cyclic. Find all the other generators.

(c) Prove that the group$(3\mathbb{Z}, \oplus)$ is cyclic and give two possible generators.

(04) (a) Let $S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$ and $A_3 = \{I, (1\ 2\ 3), (1\ 3\ 2)\}$. Prove that A_3 is a normal subgroup of S_3.

(b) Let $G = \{f_{ab} | a, b \in \mathbb{R}, a \neq 0\}$ be a group under composition mapping. Define $f: \mathbb{R} \to \mathbb{R}$ by $f_{ab}(x) = ax + b$. Let $N = \{f_{1b} | b \in \mathbb{R}\}$ be a subset of G. Prove that N is a normal subgroup of G.

(05) (a) Let H and K be subgroups of a group G of orders 24 and 36 respectively. Find the possible orders of $H \cap K$.

(b) Let G be a cyclic group of order 12 generated by a. Let H and K be the subgroups generated by a^3 and a^4 respectively. Prove that $o(H) = 4, o(K) = 3, G = HK$ and $H \cap K = \{e\}$.

(C) Determine whether $\mathbb{Z}_{18}$ has a subgroup of order 5.

(06) (a) Determine which of the following mappings $\phi: \mathbb{C}^\times \to \mathbb{R}^\times$ are homomorphisms under multiplication and which are not.

(i) $f(z) = |z|$ (iii) $f(z) = 2|z|$
(ii) $f(z) = 1 + |z|$ (iv) $f(z) = |z|^2$.

(b) Given a map $\phi: (\mathbb{R}^\times, .) \to (\mathbb{R}, +)$ by $\phi(x) = x$. Prove that $(\mathbb{R}^\times, .)$ Is not isomorphic to $(\mathbb{R}, +)$.