

THE OPEN UNIVERSITY OF SRI LANKA
Faculty of Engineering Technology
Department of Mathematics & Philosophy of Engineering



Bachelor of Science Honours in Engineering /
Bachelor of Software Engineering Honours

Final Examination (2023/2024)
MHZ5355 / MHZ5375: Discrete Mathematics

Date: 31st January 2025 (Friday)

Time: 14:00 – 17:00

Instruction:

- Answer only five questions.
- Please answer a total of five (05) questions, choosing at least one from each single section.
- All symbols are in standard notation and state any assumption that you made.

SECTION – A

Q1.

- i. Using the Principle of Mathematical Induction, prove that the following statements.
 - a) $6^n - 1$ is divisible by 5 for all integers $n \geq 1$. [20%]
 - b) $1 + 4 + 7 + \dots + (3n - 2) = \frac{n(3n-1)}{2}$ for all integers $n \geq 1$. [20%]
- ii. Let $a, b, c \in \mathbb{Z}^+$. If $a|b$ and $a|c$, prove that,
 - a) $a|b + 2c$. [10%]
 - b) $a|3b - 4c$ [10%]
- iii.
 - a) Compute the greatest common divisor (gcd) of 858 and 253 using the Euclidean algorithm. [10%]
 - b) Determine all integer solutions to the following Diophantine equation:
 $858x_0 + 253y_0 = 33$. [20%]
 - c) Find the least common multiple (lcm) of 858 and 253. [10%]

Q2.

i.

- a) Let m be a positive integer and a, b, c and d are integers. Show that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$ and $ac \equiv bd \pmod{m}$. [15%]
- b) Suppose a and b are integers such that $a \equiv 4 \pmod{13}$ and $b \equiv 9 \pmod{13}$. Find the integer c where $0 \leq c \leq 12$ such that $c \equiv 2a + 3b \pmod{13}$. [20%]

ii. Let $\gcd(a, b) = 1$. Show that $\gcd(2a + b, a + 2b) = 1$ or 3 . [25%]

iii. Solve the following system of congruence using the Chinese Remainder Theorem.

$$\begin{aligned} 2x &\equiv 1 \pmod{3} \\ 4x &\equiv 3 \pmod{5} \\ 5x &\equiv 2 \pmod{7}. \end{aligned} \quad [40\%]$$

SECTION – B

Q3.

i. Determine whether the operations " $*$ " : $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R}$ defined below are binary operation or not. If " $*$ " is a binary operation, prove your answer. If not, explain the reason. [30%]

- a) $x * y = \sqrt{xy}$; $x, y \in \mathbb{Z}$,
b) $x * y = x^2 - y^2$; $x, y \in \mathbb{Z}$,
c) $x * y = \frac{x+y}{x-y}$; $x, y \in \mathbb{Z}$ and $x \neq y$.

ii. Define a Group in usual notation. [10%]

Let $*$ be the operation defined by $x * y = |x - y| + xy$. Here $x, y \in \mathbb{N}$.

- a) Show that the operation $*$ is commutative. [15%]
b) Is the operation Associative? Justify your answer. [15%]

- iii. Consider the binary operation " $*$ " defined on the set $A = \{a, b, c, d\}$ by the following table:

$*$	a	b	c	d
a	c	d	a	b
b	d	a	b	c
c	a	b	c	d
d	b	c	d	a

- a) Find the identity element of the group and the inverse of each element. [15%]
b) Solve the equation $b * x = d$ in the group. Is this group abelian? [15%]

Q4.

- i. Define a semi-group $(G, \#)$ in usual notation.
Let operations " $\#$ " : $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, defined as follows.

- a) $a \# b = a^b$ for all $a, b \in \mathbb{R}$;
b) $c \# d = 3(c + d)$ for all $c, d \in \mathbb{R}$;
c) $e \# f = \sqrt{ef}$ for all $e, f \in \mathbb{R}^+$.

Verify whether $(\mathbb{R}, \#)$ is a semi-group or not for each of the above cases.

[45%]

- ii. Let $\mathbb{Z}_{12} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ be a group under $\oplus_{12}$.
Define $f : \mathbb{Z}_{12} \rightarrow \mathbb{Z}_{12}$ such that $f(x) = 3x$ and f is a homomorphism. Find $\text{Ker } f$.
[25%]

- iii. Show that $\mathbb{R}$, under addition, is isomorphic to $\mathbb{R}^+$ under multiplication, using the function $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ given by $\phi(x) = 3^x$.
[30%]

SECTION – C

Q5.

- i. By drawing each of the following graphs, determine whether they are simple graphs or not. Provide a clear explanation to justify your answer.

a) $V(G_1) = \{v_i | i = 1, 2, 3, 4, 5\}$ and $E(G_1) = \{\{v_i, v_j\} | i < j\}$ [20%]

b) $V(G_2) = \{\frac{\pi}{n} | n = 1, 2, 3, 4, 5, 6\}$ and $E(G_2) = \{\{\frac{\pi}{n}, \frac{\pi}{m}\} | \sin(\frac{\pi}{n}) \leq \cos(\frac{\pi}{m})\}$ [20%]

- ii. State whether the following statements are true or false. Justify your answer.

a) There exists a graph of 14 edges with two vertices of degree 3, three vertices of degree 2 and two vertices of degree 1. [15%]

b) If G is a graph with $|V(G)| = 32$ and $|E(G)| = 496$. Then G can be a complete graph. [15%]

- iii. Let G be a graph whose adjacency matrix A is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

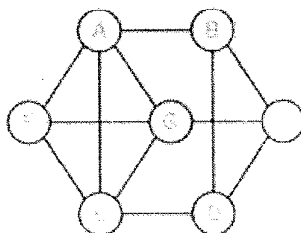
a) Without drawing a graph of G , explain whether G is connected or not. [15%]

b) If $V(G) = \{v_1, v_2, v_3, v_4\}$, then find the number of different paths (walks) of length three (3) joining vertices v_1 and v_3 . [10%]

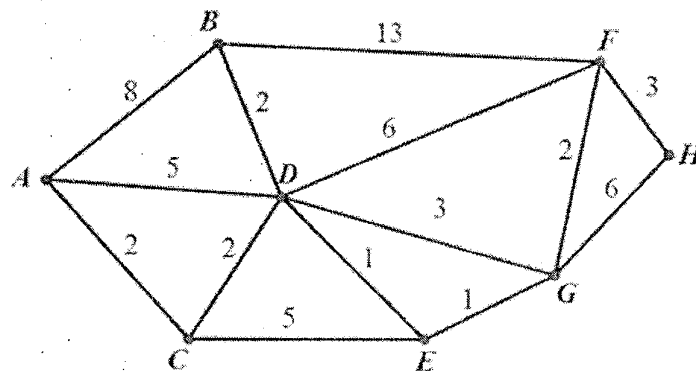
c) Draw the graph of an adjacency matrix A . [05%]

Q6.

- i. Determine whether the following graph is Eulerian, Hamiltonian or both. [20%]



- ii. A tree has 100 leaves, 20 vertices of degree 6 and half of the remaining vertices have degree 4. The left-over vertices are degree 2, how many vertices are of degree 2? [20%]
- iii. Use Dijkstra's algorithm to find the shortest route between node A and every other node in the following network. [60%]



SECTION – D

Q7.

- i. Iterate the Eco-system growth given by the relation $t_{n+1} - \lambda t_n = 0$ starting with $t_0 = 0.3$, taking $\lambda = 0.5$, and draw the diagram. Hence deduce t_n as $n \rightarrow \infty$. (At least 5 iteration steps are necessary) [20%]
- ii. Consider the relationship $x_{n+1} = \cos x_n$, with the initial condition of $x_0 = \pi/2$, obtain the convergent value of the model. (Up to 4 decimal places probably) [20%]
- iii. Consider the iteration given by $Z_{n+1} = Z_n^2$ with $Z_0 = 1 + 3i$. Find the values of Z_1 to Z_3 . Discuss the solution Z_n as $n \rightarrow \infty$. [10%]
- iv. Suppose a system with two unknowns x and y , is modeled in the form of a system of differential equations

$$\begin{aligned} \frac{dx}{dt} - 2x + 6y &= 0 \\ \frac{dy}{dt} - x - 7y &= 0 \end{aligned}$$

with initial conditions $x = 1$ and $y = 0$ when $t = 0$.

Find the space values (x_t, y_t) , when $t = 1$ and 2 .

[50%]

Q8.

- i. Let $L = \{1, 19, 91\}$ and $M = \{91, 919, 191\}$ be languages. Find the concatenations LM and ML . [20%]
- ii. Suppose $L(G) = \{a^m b^n \mid m > 0 \text{ and } n \geq 0\}$. Find out the grammar G which produces L . [20%]
- iii. Show that the string $\{(x + y) * x - z * y / (x + x)\}$ is a sentence generated by the grammar and starting symbol S and production P . [20%]
 $P = \{S \rightarrow x, S \rightarrow y, S \rightarrow z, S \rightarrow S + S, S \rightarrow S - S, S \rightarrow S * S, S \rightarrow S / S, S \rightarrow (S)\}$.
- iv. Let $M = \{S, I, \delta, S_0, F\}$ be a Non-Deterministic Finite Automata (NFA), where S is a finite set of states, I is a finite set of input symbols, δ is the transition function, S_0 is the initial state, and F is the set of final states.
 Transition Table for the above Non-Deterministic Finite Automata as follows:

States	Inputs	
	a	b
0	{0,1}	{0,4}
1	{1}	{2}
2	{2,3}	-
3	{3}	{3}
4	{5}	-
5	{5}	{4,5}

The initial state is **0**, and the set of final states are **{3, 5}**.

- a) Depict the finite automaton's transition graph. [10%]
- b) Show that the string **ababab** is accepted by the Non-deterministic Finite Automaton by applying the transition function. [30%]

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