



The Open University of Sri Lanka

B.Sc./B.Ed., Continuing Education Degree Programme

Final Examination-2025/2026

ADU5302/ADE5302-Mathematical Methods

Applied Mathematics -Level 05

DURATION: Two (02) Hours

Date: 07.01.2026

Time: 1.30p.m. - 3.30p.m.

Important: Answer FOUR questions only. Select at least ONE question from part A.

Part A

1. (a) Find the characteristic values and characteristic functions for the following Sturm-Liouville problem:

$$\frac{d^2 y}{dx^2} + \lambda y = 0, \quad y(0) = 0 \text{ and } y(1) = 0..$$

- (b) Show that the set of characteristic functions are mutually orthogonal in the interval $0 \leq x \leq 1$.

2. The Rodrigue's formula for the n^{th} degree Legendre polynomial denoted by $P_n(x)$ is given as

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

$P_n(x)$ is also given by the sum

$$P_n(x) = \sum_{m=0}^M \frac{(-1)^m (2n-2m)!}{2^n m! (n-m)! (n-2m)!} x^{n-2m}, \quad n = 0, 1, 2, \dots$$

where $M = \frac{n}{2}$ or $\frac{n-1}{2}$, whichever is an integer.

- (a) Prove that $(1-x^2)p'_{n-1} = n(xp_{n-1} - p_n)$.
 (b) Find $p_0(x)$, $p_1(x)$, $p_2(x)$ and $p_3(x)$.
 (c) Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre polynomials.
 (d) Show that

$$\int_{-1}^1 p_n(x)(1-2xt+t^2)^{-\frac{1}{2}} dx = \frac{2t^n}{2n+1}$$

where n is a positive integer.

$$(\text{Hint: } (1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n \cdot p_n(x))$$

(You may use the following recurrence relations, if necessary, without proof.)

$$\begin{aligned} P'_n(x) &= xP'_{n-1}(x) + nP_{n-1}(x), \\ P'_{n+1}(x) - P'_{n-1}(x) &= (2n+1)P_n(x), \\ (n+1)p_{n+1} &= (2n+1)xp_n - np_{n-1}, \\ (x^2-1)(p'_n) &= n(xp_n - p_{n-1}). \end{aligned}$$

Part B

1.(a) Find the Laplace transform $L(t)$ of $\cos at \sinh at$.

(b) Find the inverse Laplace transform of the function $F(s) = \frac{2-3s}{(s-2)(s^2+3)}$.

(c) Use the convolution theorem to find the inverse Laplace transform of

$$\frac{1}{(s+1)(s+9)^2}.$$

(d) Solve the following boundary value problem using the Laplace transform method:

$$\frac{d^2 y}{dt^2} - 3 \frac{dy}{dt} + 2y = 4 \quad \text{subject to } y(0) = 2, \quad y'(0) = 3.$$

2. (a) Given that $f(x) = \begin{cases} \alpha x; & 0 \leq x \leq 1 \\ \alpha; & 1 < x \leq 2 \end{cases}$ where $\alpha > 0$.

Find the Fourier series expansion of $f(x)$ in the interval $0 \leq x \leq 2$.

- (b) Obtain the cosine series for $f(x) = (x-2)^2$ in the interval $(0, 2)$.

3. (a) The Gamma function denoted by $\Gamma(p)$ corresponding to the parameter p is defined by the

improper integral $\Gamma(p) = \int_0^{\infty} e^{-t} t^{p-1} dt$, ($p > 0$). Prove each of the following:

(i)
$$\int_0^{\infty} x^2 e^{-x^4} dx \times \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{4\sqrt{2}}.$$

(ii)
$$\int_0^{\infty} \frac{x^4}{4^x} dx = \frac{\Gamma(5)}{(\log 4)^5}.$$

- (b) The Beta function denoted by $\beta(p, q)$ is defined by $\beta(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx$,

where p and q are positive parameters.

- (i) Use Gamma function and Beta function to evaluate the following integral:

$$\int_0^1 \frac{1}{\sqrt{1-x^4}} dx.$$

- (ii) Prove that
$$\int_0^{\infty} \frac{1}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

4. Let $J_p(x)$ be the Bessel function of order p given by the expansion

$$J_p(x) = x^p \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+p} m! \Gamma(p+m+1)}.$$

(a) Prove each of the following results:

$$(i) \frac{d}{dx} \{x^p J_p(x)\} = x^p J_{p-1}(x).$$

$$(ii) J_n'' = \frac{1}{4} [J_{n-2} - 2J_n + J_{n+2}]$$

(b) Find a and c so that $J_2(x) - J_0(x) = aJ_c''(x)$.

(Hint: You may use the following recurrence relations, if necessary, without proof.)

$$\frac{d}{dx} \{x^{-p} J_p(x)\} = -x^{-p} J_{p+1}(x),$$

$$\frac{d}{dx} \{J_p(x)\} = J_{p-1}(x) - \frac{p}{x} J_p(x),$$

$$J_p'(x) = \frac{p}{x} J_p(x) - J_{p+1}(x),$$

$$J_p'(x) = \frac{1}{2} \{J_{p-1}(x) - J_{p+1}(x)\},$$

$$J_{p-1}(x) + J_{p+1}(x) = \frac{2p}{x} J_p(x).$$