The Open University of Sri Lanka

B.Sc./ B. Ed. Degree programme

Final Examination 2025/2026

Level 05 - Pure Mathematics

PEU5300 - Riemann Integration

Date: 2026/01/12 **Time: 1.30pm to 3.30pm**

Instructions

- This paper contains 06 questions from Page 1 to Page 4.
- Answer **four (04)** questions.
- Total maximum mark obtainable is 100. The marks assigned for each question and parts thereof are indicated in square brackets.

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1. (a) Let $f(x) = k$ be a constant function define on $[a, b]$. Using the upper and lower Riemann integrals, prove that f is Riemann integrable on $[a, b]$. [7 marks]

   (b) Let $f : [1, 4] \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 3, & x \in [1, 3] \\ 4, & x \in (3, 4]. \end{cases}$$

   i. Show that $U(P, f) \geq 10$ for any partition P of $[1, 4]$. [10 marks]

   ii. Prove that the upper Riemann integral $\overline{\int_1^4} f(x)\, dx = 10$. [8 marks]

2. (a) Let f be a bounded function on $[a, b]$. For each $\epsilon > 0$ if there exists a partition P_ϵ of $[a, b]$ such that

$$U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon,$$

   then prove that f is Riemann integrable. [8 marks]

(b) Let f be a bounded function on $[a,b]$ so that $|f(x)| \leq l$ for all $x \in [a,b]$, where $l > 0$.

i. For any partition P on $[a,b]$, show that

$$U\left(P, f^2\right) - L\left(P, f^2\right) \leq 2l\left[U(P,f) - L(P,f)\right].$$

Hint: $f(x)^2 - f(y)^2 = [f(x) - f(y)]\,[f(x) + f(y)]$ [10 marks]

ii. Show that if f is integrable on $[a,b]$, then f^2 also is integrable on $[a,b]$. [7 marks]

3. (a) Let $f : [a,b] \to \mathbb{R}$ be a bounded function. If there is a sequence of partitions $\{P_n\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\lim_{n\to\infty} \left[U(P_n, f) - L(P_n, f)\right] = 0$$

then show f is Riemann integrable and $\int_a^b f(x)\,dx = \lim_{n\to\infty} U(P_n, f)$. [10 marks]

(b) Prove that a continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable. [10 marks]

(c) Compute $\int_2^3 x^x (1 + \ln x)\,dx$. Justify your answer.
Hint: Find the derivative of $f(x) = x^x$. [5 marks]

4. (a) i. State the First version of Fundamental theorem of Calculus. [3 marks]

ii. Use part (i) to calculate

$$\frac{d}{dx}\left[\int_{\sqrt{x}}^{x^2} \tan t\,dt\right]$$

for any $x \in [4, 10]$. Justify each step. [8 marks]

(b) i. State the First and Second versions of Substitution Theorem for integrals. [6 marks]

ii. Evaluate the following integral by justifying each step.

$$\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$$

[8 marks]

5. (a) Suppose that f and g are integrable functions on $[a,b]$. If $f(x) \leq g(x)$ for any $x \in [a,b]$ then show that,

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

[7 marks]

(b) Clearly stating any results you use, show that

$$\left| \int_{-2\pi}^{2\pi} x^2 \sin^8\left(e^{x^2}\right)\,dx \right| \leq \frac{16\pi^3}{3}.$$

[10 marks]

(c) Using the properties of Riemann Sums, find an integral of a suitable function over an appropriate interval that is represented by

$$\lim_{n\to\infty} \frac{2}{n} \sum_{i=1}^{n} \sin\left(\frac{2i}{n}\right).$$

Hence, find the above limit. [8 marks]

6. (a) Find and justify values of $b > 0$ that makes $\int_0^b \frac{1}{x^2-4}\,dx$ an improper integral. [4 marks]

(b) Explain why it is not possible to draw any conclusions about the convergence of $\int_1^\infty \frac{e^x}{x}\,dx$ by comparing with the integral $\int_1^\infty \frac{1}{x}\,dx$. [7 marks]

(c) Give reasons for the following integrals becoming improper integrals. Determine if each improper integrals converges and, if so, find its value justifying your answer.

i. $\int_4^5 \frac{x-2}{\sqrt{x}(x-4)}\,dx$ [7 marks]

ii. $\int_0^2 \frac{1}{\sqrt{2-x}}\,dx$ [7 marks]

----------End of the question paper----------