THE OPEN UNIVERSITY OF SRI LANKA

B.SC. DEGREE PROGRAMME

FINAL EXAMINATION 2025/2026

LEVEL 05 – PURE MATHEMATICS

PEU5304- INTRODUCTION TO COMPLEX ANALYSIS

Date: **28/11/2025** **Time:02.00pm-04.00pm**

Instructions

- This paper consists of six questions from page 01 to page 03.
- Answer **four** of six questions.
- You are permitted to use nonprogrammable calculator.

01.

(a) Prove or disprove each of the following statements.

(i) If $\operatorname{Im} z > 0$ then $\operatorname{Im} z^{-1} < 0$ for any nonzero complex numbers.

(ii) Let z_1, z_2 be two complex numbers such that $|z_1| = 2$ and $|z_2| = 5$. Then the maximum value of $|z_1^2 - 3z_2 + 5|$ is 25.

(iii) There exists z_1, z_2 complex numbers such that $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2)$ and $\operatorname{Im}(z_1 z_2) = \operatorname{Im}(z_1)\operatorname{Im}(z_2)$.

(b) Let z and w be two distinct complex numbers such that $|z| = 1$ and $\bar{z}w \neq 1$.

Prove that $\left|\frac{z-w}{1-\bar{z}w}\right| = 1$.

[Hint: For any complex number z, $|z|^2 = z.\bar{z}$]

02. (a) Let z be any complex number.

Prove that if $\arg(z-1) = \frac{\pi}{6}$ and $\arg(z+1) = \frac{2\pi}{3}$ then $|z| = 1$.

(b) Suppose $z_1, z_2 \in \mathbb{C}$ and $z_1 \neq z_2$ such that $|z_1 + \bar{z_2}| = |z_1 - \bar{z_2}|$. Show that $\arg(z_1) + \arg(z_2) = \frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$ or $\arg(z_1) + \arg(z_2) = -\frac{\pi}{2} + 2n\pi, n \in \mathbb{Z}$.

(c) Express $\frac{1-i}{1-\sqrt{3}i}$ in the polar form.

03. (a) Find the locus of the points in the complex plane that represents the complex numbers z such that $|z-1| = 2|z+2|$. Also sketch it on the argand plane.

(b) The complex number $z = x + iy$ satisfies the relation $2 \leq |z-2-3i| < 3$.

(i) Shade accurately in an argand plane, the region represented by the above relation.

(ii) Determine whether the point that is represented by $z = 4 + i$ lies inside or outside this region.

(c) Verify that $2 + i$ is a root of the polynomial $p(z) = z^4 - 3z^3 + 3z^2 - 3z + 10$.
Find all the other roots of $p(z) = 0$.

04. (a) State and prove the De Moivre's theorem for all positive integers.

(b) Let $z_1 = 1 - i$ and $z_2 = 1 + \sqrt{3}i$.

(i) Express z_1 and z_2 in polar forms.

(ii) Hence evaluate the product $z_1 z_2$.

(iii) Show that $(z_1 z_2)^{2025} = 2^{3037}(-1 + i)$.

(c) Find the cube roots of the complex number $(1 + i)$.

05. (a) Define the complex exponential function e^z for any $z \in \mathbb{C}$.

(b) Let z_1 and z_2 be two arbitrary complex numbers.
Prove that $\exp(z_1) = -\exp(z_2)$ if and only if $z_1 = z_2 + (2k+1)\pi i, k \in \mathbb{Z}$.
Deduce that $e^z = -1$ if and only if $z = (2k+1)\pi i, k \in \mathbb{Z}$.

(c) For any complex number z, show that $\sinh z = -i \sin iz$.
Hence show that $\sinh z = 0$ if and only if $z = in\pi, n \in \mathbb{Z}$.

06. (a) Define the logarithmic function $\log z$, for any nonzero $z \in \mathbb{C}$.
Find the roots of the equation $\log z = \left(\frac{121\pi}{2}\right) i$.

(b) Let z be a nonzero complex number and r is an irrational number. Define z^r.
Evaluate $(1+i)^{\pi}$.

(c) Prove or disprove each of the following.

(i) For each $n_1, n_2 \in \mathbb{N}$ if $n_1 \neq n_2$ then $e^{i\sqrt{3}(\frac{\pi}{6}+2n_1\pi)} = e^{i\sqrt{3}(\frac{\pi}{6}+2n_2\pi)}$.

(ii) Suppose α is a given irrational number. Then
Principal value of $(z_1 z_2)^{\alpha}$ = (Principal value of z_1^{α}). (Principal value of z_2^{α}), for each $z_1, z_2 \in \mathbb{C}$.

End of the Question Paper

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