

The Open University of Sri Lanka

B.Sc/B.Ed. Degree Programme

Final Examination - 2025/2026

Pure Mathematics - Level 05

PEU5305 – Complex Analysis I

Duration: Two hours



Date: 19.01.2026

Time: 09.30 a.m. – 11.30 a.m.

Answer **FOUR** Questions **ONLY**.

1.

- a) Determine where the function $f(z) = (x^2y - y^3) + i(x^3 - 3xy^2)$ is differentiable and where it is analytic.
- b) Let $u(x, y) = \cosh x \cos y$.
 - i. Prove that the function $u(x, y)$ is harmonic.
 - ii. Find a function $v(x, y)$ such that $f(z) = u + iv$ is analytic in $\mathbb{C}$, and express $f(z)$ in terms of z .
- c) Let $f(z) = u + iv$ be analytic in a region G . Show that if $u + 3v = 7$ in G , then $f(z)$ is constant in G .

2.

- a) Prove each of the following identities:
 - i. $e^{-i\theta} = \cos \theta - i \sin \theta$, for all $\theta \in \mathbb{R}$.
 - ii. $e^{z_1 - z_2} = \frac{e^{z_1}}{e^{z_2}}$, for all $z_1, z_2 \in \mathbb{C}$.
 - iii. $\cos^2 z + \sin^2 z = 1$, for all $z \in \mathbb{C}$.
- b) Solve the equation $e^{2z} = i$.
- c) Prove that $\sin z = 0$ if and only if $z = n\pi$, $n \in \mathbb{Z}$.

3.

a) State Cauchy's Integral Formula.

b) Using Cauchy's Integral Formula, evaluate the integral $\int_C \frac{e^z}{z-1} dz$, where C is the circle with radius 3 centered at $1 + i$, oriented counterclockwise.

c) Apply Cauchy's Integral Formula to $\cos z$ with a suitable contour to show that

$$\int_0^{2\pi} \cosh(\cos \theta) \sin(\sin \theta) d\theta = 0.$$

4.

a) State Green's Theorem in the plane.

b) Evaluate the line integral $\int_C (3x + 4y)dx + (-2x + 6y) dy$, where C is the unit circle given by $x = \cos \theta$, $y = \sin \theta$; $0 \leq \theta \leq 2\pi$.

i. Using Green's Theorem in the plane.

ii. Using the parameterization of C .

c) Let A be the area bounded by a simple closed contour C .

$$\text{Show that } A = \frac{1}{2} \int_C (x dy - y dx).$$

Using the above formula, find the area enclosed by the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, where $\alpha, \beta > 0$.

5.

a) Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{z^n}{n+1}$.

b) Let $f(z) = \frac{1}{(2z-1)(z+2)}$. Find the Laurent series expansion of f in the following annulus $\frac{1}{2} < |z| < 2$.

c) Show that the function $g(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$.

d) Find and classify the singularities of the function $h(z) = \frac{e^{2z}}{(z-1)^2(z+2i)}$.

6.

a) State Cauchy's Residue Theorem.

b) Using Cauchy's Residue Theorem, calculate each of the following contour integrals:

i. $\int_C \operatorname{cosec} z \, dz$, where C is the circle $|z| = 5$, oriented counterclockwise.

ii. $\int_C \frac{2z}{z^2+4} \, dz$, where C is the circle $|z| = 3$, oriented counterclockwise.

c) Using Cauchy's Residue Theorem, evaluate the integral $\int_0^{2\pi} \frac{1}{4+3 \cos \theta} \, d\theta$.