MPZ 3230 – Assignment No. 01
Academic Year – 2007

(01) (i) Given $\underline{a}=\underline{i}+2\underline{j}-3\underline{k}$ and $\underline{b}=3\underline{i}+\underline{j}+2\underline{k}$

(a) Show that the vectors $\underline{a}+\underline{b}$ and $\underline{a}-\underline{b}$ are mutually orthogonal.
(b) Calculate the acute angle between the vectors $2\underline{a}+\underline{b}$ and $\underline{a}+2\underline{b}$
(c) Obtain a unit vector orthogonal to both $\underline{a}$ and $\underline{b}$

(ii) If $\underline{a}=\underline{i}+2\underline{j}-3\underline{k}$, $\underline{b}=2\underline{i}-\underline{j}+\underline{k}$ and $\underline{c}=-\underline{i}+4\underline{j}-\underline{k}$ then calculate

(a) $\underline{a}.\underline{b}\times\underline{c}$
(b) $(\underline{a}\times\underline{b})\times\underline{c}$
(c) $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{a})$
(d) $(\underline{a}\times\underline{b})\times\{(\underline{b}\times\underline{c})\times(\underline{c}\times\underline{a})\}$

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(iii) Suppose $\underline{a},\underline{b}$ and $\underline{c}$ are represented by edges of tetrahedron. How do you can interpret value of $\frac{1}{6}(\underline{a}\wedge\underline{b}).\underline{c}$

(02) (i) Expand (a) $(\underline{a}\times\underline{b}).(\underline{b}\times\underline{c})$
(b) $(\underline{a}\times\underline{b})\times(\underline{b}\times\underline{c})$
(c) $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})$ in different ways and express $\underline{d}$ in terms of $\underline{a},\underline{b}$ and $\underline{c}$

(ii) In the position vector of a point at time t is $\underline{R}(t)=\sin t\underline{i}+t\underline{j}+\cos t\underline{k}$ determine the time at which the velocity vector and the acceleration vector become perpendicular to each other.

(iii) Show that $\frac{d}{dt}(\underline{a}\times\underline{b}\times\underline{c})=\frac{d\underline{a}}{dt}\times\underline{b}\times\underline{c}+\underline{a}\times\frac{d\underline{b}}{dt}\times\underline{c}+\underline{a}\times\underline{b}\times\frac{d\underline{c}}{dt}$

(iv) If n, a and b are constant and $\underline{p}=a\cos nt\underline{i}+b\sin nt\underline{j}$, prove that $\underline{p}\times\frac{d\underline{p}}{dt}=nab\underline{k}$

(03) (i) Radium decomposes at a rate proportional to the amount present. If of 100 mg set aside now there will be left 96 mg 100 years hence, find how much will be left t centuries from the time when the radium was set aside, how much will be left after 2.58 centuries, and the half-life of radium.

(ii) Find the general solution of $a\left[x\frac{dy}{dx}+2y\right]=xy\frac{dy}{dx}$ and then find a particular solution in which y = a when x = 2a.

(04) Solve the following differential equations.

(i) $(x+y)dx+(x+y-2)dy=0$
(ii) $(4x+xy^2)dx+(y+x^2y)dy=0$
(iii) $xy(xdy+ydx)=6y^3dy$
(iv) $xy\,dy=(y+1)(1-x)dx$
(v) $(x^2+y^2)dx+2xy\,dy=0$

(05) Determine whether or not each of equations exact. If it is exact or not find the solution by using suitable method.

(i) $(\cos y+y\cos t)dt+(\sin t-t\sin y)dy=0$
(ii) $(3x^2+y^2)dx-2xy\,dy=0$
(iii) $\frac{xy-1}{x^2y}dx-\frac{1}{xy^2}dy=0$
(iv) $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$
(v) $(x+y)dx-(x-y)dy=0$
(vi) $x^2\frac{dy}{dx}+xy+\sqrt{1-x^2-y^2}=0$

Please send your assignment **on or before due date according to activity diary** to the following address.
Please send your answer with your address (write back of your answer sheet) and when you are doing assignment use both sides of paper.

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MPZ 3230
Model answer No. 01 – Academic Year 2007

(01) (i) Given $\underline{a} = \underline{i} + 2\underline{j} - 3\underline{k}$

$\underline{b} = 3\underline{i} + \underline{j} + 2\underline{k}$

(a) $\underline{a} + \underline{b} = (\underline{i} + 2\underline{j} - 3\underline{k}) + (3\underline{i} + \underline{j} + 2\underline{k})$

$= 4\underline{i} + 3\underline{j} - \underline{k}$

$\underline{a} - \underline{b} = (\underline{i} + 2\underline{j} - 3\underline{k}) - (3\underline{i} + \underline{j} + 2\underline{k})$

$= -2\underline{i} + \underline{j} - 5\underline{k}$

If $(\underline{a} + \underline{b})$ and $(\underline{a} - \underline{b})$ mutually orthogonal

$(\underline{a} + \underline{b}).(\underline{a} - \underline{b})$ must be zero.

$(\underline{a} + \underline{b}).(a - b) = (4\underline{i} + 3\underline{j} - \underline{k}).(-2\underline{i} + \underline{j} - 5\underline{k})$

$= -8 + 3 + 5$

$= 0$

There for $(\underline{a} + \underline{b})$ and $(\underline{a} - \underline{b})$ are mutually orthogonal.

(b) $2\underline{a} + \underline{b} = 2(\underline{i} + 2\underline{j} - 3\underline{k}) + 3\underline{i} + \underline{j} + 2\underline{k}$

$= 5\underline{i} + 5\underline{j} - 4\underline{k}$

$\underline{a} + 2\underline{b} = (\underline{i} + 2\underline{j} - 3\underline{k}) + 2(3\underline{i} + \underline{j} + 2\underline{k})$

$= 7\underline{i} + 4\underline{j} - \underline{k}$

Acute angle between $(2\underline{a} + \underline{b})$ and $(\underline{a} + 2\underline{b})$ is θ

$(2\underline{a} + \underline{b}).(\underline{a} + 2\underline{b}) = |2\underline{a} + \underline{b}||\underline{a} + 2\underline{b}| \cos\theta$

$$\cos\theta = \frac{(2\underline{a} + \underline{b}).(\underline{a} + 2\underline{b})}{|2\underline{a} + \underline{b}||\underline{a} + 2\underline{b}|} \quad \text{---- (A)}$$

Find $(2\underline{a} + \underline{b}).(\underline{a} + 2\underline{b}) = (5\underline{i} + 5\underline{j} - 4\underline{k}).(7\underline{i} + 4\underline{j} + \underline{k})$

$= 51$

Find $|2\underline{a} + \underline{b}| = \sqrt{25 + 25 + 16} = \sqrt{66}$

$|\underline{a} + 2\underline{b}| = \sqrt{49 + 16 + 1} = \sqrt{66}$

Then substitute A; $\cos\theta = \dfrac{51}{\sqrt{66}\sqrt{66}}$

$$\theta = \cos^{-1}\left(\frac{51}{66}\right)$$

$$= 39.4^0$$

(c) The unit vector orthogonal to both $\underline{a}$ and $\underline{b}$ is $\dfrac{\underline{a}\times\underline{b}}{|\underline{a}\times\underline{b}|}$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -3 \\ 3 & 1 & 2 \end{vmatrix}$$

$$= \underline{i}(4-(-3)) - \underline{j}(2-(-9)) + \underline{k}(1-6)$$

$$= 7\underline{i} = 11\underline{j} - 5\underline{k}$$

$$|\underline{a}\times\underline{b}| = \sqrt{49+121+25}$$

$$= \sqrt{195}$$

Unit vector orthogonal both $\underline{a}$ and $\underline{b}$ is $\dfrac{1}{\sqrt{195}}(7\underline{i} - 11\underline{j} - 5\underline{k})$

(ii) $\underline{a} = \underline{i} + 2\underline{j} - 3\underline{k}$

$\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$

$\underline{c} = -\underline{i} + 4\underline{j} - \underline{k}$

(a) $\underline{a}.(\underline{b}\times\underline{c})$

$$(\underline{b}\times\underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -1 & 4 & -1 \end{vmatrix}$$

$$= \underline{i}(1-4) - \underline{j}(-2-(1)) + \underline{k}(8-1)$$

$$= -3\underline{i} + \underline{j} + 7\underline{k}$$

$$\underline{a}.\underline{b}\times\underline{c} = (\underline{i} + 2\underline{j} - 3\underline{k}).(-3\underline{i} + \underline{j} + 7\underline{k})$$

$$= -3 + 2 - 21$$

$$= (-22)$$

(b) $(\underline{a}\times\underline{b})\times\underline{c}$

$$(\underline{a}\times\underline{b}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= \underline{i}(2-3)-\underline{j}(1-(-6))-\underline{k}(-1-4)$$
$$= -1\underline{i}-7\underline{j}-5\underline{k}$$

$$(\underline{a}\times\underline{b})\times\underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -7 & -5 \\ -1 & 4 & -1 \end{vmatrix}$$
$$= \underline{i}(7-(-20))-\underline{j}(1-5)+\underline{k}(-4-7)$$
$$= 27\underline{i}+4\underline{j}-11\underline{k}$$

(c) $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{a})$

$$(\underline{a}\times\underline{b}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -3 \\ 2 & -1 & 1 \end{vmatrix}$$
$$= \underline{i}(2-3)-\underline{j}(1-(-6))+\underline{k}(-1-4)$$
$$= -\underline{i}-7\underline{j}-5\underline{k}$$

$$(\underline{c}\times\underline{a}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & 4 & -1 \\ 1 & 2 & -3 \end{vmatrix}.$$
$$= \underline{i}(-12-(-2))-\underline{j}(3-(-1))+\underline{k}(-2-4)$$
$$= -10\underline{i}-4\underline{j}-6\underline{k}$$

$$(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{b}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -1 & -7 & -5 \\ -10 & -4 & -6 \end{vmatrix}$$
$$= \underline{i}(42-20)-\underline{j}(6-50)+\underline{k}(4-70)$$
$$= 22\underline{i}+44\underline{j}-66\underline{k}$$

(d) $(\underline{a}\times\underline{b})\times\{(\underline{b}\times\underline{c})\times(\underline{c}\times\underline{a})\}$

$$(\underline{a}\times\underline{b}) = -\underline{i}-7\underline{j}-5\underline{k}$$

$$(\underline{b}\times\underline{c}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & -1 & 1 \\ -1 & 4 & -1 \end{vmatrix}$$
$$= \underline{i}(1-4)-\underline{j}(-2-(-1))+\underline{k}(8-1)$$
$$= -3\underline{i}+\underline{j}+7\underline{k}$$

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$$(\underline{c}\times\underline{a})=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & 4 & -1\\ 1 & 2 & -3\end{vmatrix}$$

$$=\underline{i}(-12-(-2))-\underline{j}(3-(-1))+\underline{k}(-2-4)$$

$$=-10\underline{i}-4\underline{j}-6\underline{k}$$

$$\{(\underline{b}\times\underline{c})\times(\underline{c}\times\underline{a})\}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -3 & 1 & 7\\ -10 & -4 & -6\end{vmatrix}$$

$$=\underline{i}(-6-(-28))-\underline{j}(18-(-70))+\underline{k}(12-(-10))$$

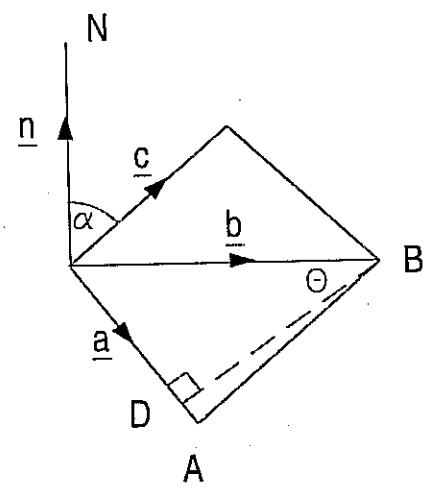
$$=22\underline{i}-88\underline{j}+22\underline{k}$$

$$(\underline{a}\times\underline{b})\times\{(\underline{b}\times\underline{c})\times(\underline{c}\times\underline{a})\}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ -1 & -7 & -5\\ 22 & -88 & 22\end{vmatrix}$$

$$=\underline{i}(-154-(-440))-\underline{j}(-22-(-110))+\underline{k}(88+154)$$

$$=-594\underline{i}-88\underline{j}+242\underline{k}$$

(iii)



Let $\underline{n}$ is the unit vector perpendicular to plan $\underline{a}$ and $\underline{b}$, and $O\hat{B}D=\theta$ and $C\hat{O}N=\underline{\alpha}$

Volume of tetrahedron is $=\frac{1}{3}(\text{area of base})\times(\text{height})$ ---- (1)

$$\text{area of base} = \frac{1}{2}\overrightarrow{OA} \times \overrightarrow{DB} \sin\theta.\underline{n}$$

$$= \frac{1}{2}a.b\sin\theta$$

$$\text{height is} \quad = \overrightarrow{CO}\cos\alpha$$
$$= \underline{c}\cos\alpha \times \hat{\underline{n}}$$
$$= \underline{c}.\hat{\underline{n}}\cos\alpha$$
$$= \underline{c}.\hat{\underline{n}}$$

Substitute equation (1) $\Rightarrow$

$$\Rightarrow \frac{1}{3}\left(\frac{1}{2}a.b\sin\theta\right) \times \underline{c}.\hat{\underline{n}}$$
$$= \frac{1}{6}(a.b\sin\theta\ n.c)$$
$$= \frac{1}{6}(a\times b).c$$

Volume of tetrahedron is $= \frac{1}{6}(a\times b).c$

(02) (i) (a) $(\underline{a}\times\underline{b}).(\underline{b}\times\underline{c})$

Let $\underline{a}\times\underline{b} = \underline{e}$

Hence; $(\underline{a}\times\underline{b}).(\underline{b}\times\underline{c}) = \underline{e}.(\underline{b}\times\underline{c})$

$= \underline{e}\times\underline{b}.\underline{c}$ (As we can interchange cross and the dot)

Now $= \left[(\underline{a}\times\underline{b})\times\underline{b}\right].\underline{c}$

Using vector triple product $= \left[(\underline{b}.\underline{a})\underline{b} - (\underline{b}.\underline{b})\underline{a}\right]\underline{c}$

$= (b.a)(\underline{b}.\underline{c}) - b^2(\underline{a}.\underline{c})$

(b) $(\underline{a}\times\underline{b})\times(\underline{b}\times\underline{c})$

Let $(\underline{b}\times\underline{c}) = \underline{e}$

Hence $(\underline{a}\times\underline{b})\times(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\times\underline{e}$

Using vector triple product $= (\underline{e}.\underline{a})b - (\underline{e}.\underline{b})\underline{a}$

$= \left[(\underline{b}\times\underline{c}).\underline{a}\right].\underline{b} - \left[(\underline{b}\times\underline{c}).\underline{b}\right].\underline{a}$

Can change triple scalar products $= (\underline{a}\times\underline{c}.\underline{b})\underline{b} - (\underline{b}\times\underline{b}.\underline{c})\underline{a}$

Let $\underline{b}\times\underline{b} = 0$ $= (\underline{a}\times\underline{b})(b.b) - 0$

$= (\underline{a}\times\underline{b}).b^2$

(c) $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})$

Let $(\underline{a}\times\underline{b})=\underline{e}$

Hence $(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})=\underline{e}\times(\underline{c}\times\underline{d})$

$=\underline{e}\times(\underline{c}\times\underline{d})$

$=(\underline{e}.\underline{d})\underline{c}-(\underline{e}.\underline{c})\underline{d}$

$=(\underline{a}\times\underline{b}.\underline{d})\underline{c}-(\underline{a}\times\underline{b}.\underline{c}).\underline{d}$ ----------(1)

$(\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d})$

Let $(\underline{c}\times\underline{b})\times(\underline{c}\times\underline{d})=(\underline{a}\times\underline{b})\times\underline{r}$

$=(\underline{r}.\underline{a})\underline{b}-(\underline{r}.\underline{b})\underline{a}$

$=(\underline{c}\times\underline{d}.\underline{a})\underline{b}-(\underline{c}\times\underline{d}.\underline{b})\underline{a}$

$(\underline{c}\times\underline{b})\times(\underline{c}\times\underline{d})=(\underline{c}\times\underline{d}.\underline{a})\underline{b}-(\underline{c}\times\underline{d}.\underline{b})\underline{a}$ ----------(2)

By (1) and (2),

$$(\underline{a}\times\underline{b}.\underline{d})\underline{c}-(\underline{a}\times\underline{b}.\underline{c}).\underline{d}=(\underline{c}\times\underline{d}.\underline{a})\underline{b}-(\underline{c}\times\underline{d}.\underline{b})\underline{a}$$

$$(\underline{a}\times\underline{b}.\underline{c}).\underline{d}=(\underline{c}\times\underline{d}.\underline{b})\underline{a}-(\underline{c}\times\underline{d}.\underline{a})\underline{b}+(\underline{a}\times\underline{b}.\underline{d})\underline{c}$$

$$\underline{d}=\frac{(\underline{c}\times\underline{d}.\underline{b})}{(\underline{a}\times\underline{b}.\underline{c})}\underline{a}-\frac{(\underline{c}\times\underline{d}.\underline{a})}{(\underline{a}\times\underline{b}.\underline{c})}\underline{b}+\frac{(\underline{a}\times\underline{b}.\underline{d})}{(\underline{a}\times\underline{b}.\underline{c})}\underline{c}$$

Then let the above equation, $\frac{(\underline{c}\times\underline{d}.\underline{b})}{(\underline{a}\times\underline{b}.\underline{c})}$, $\frac{(\underline{c}\times\underline{d}.\underline{a})}{(\underline{a}\times\underline{b}.\underline{c})}$, $\frac{(\underline{a}\times\underline{b}.\underline{d})}{(\underline{a}\times\underline{b}.\underline{c})}$ are values,

There for,

Let, $P=\frac{(\underline{c}\times\underline{d}.\underline{b})}{(\underline{a}\times\underline{b}.\underline{c})}$ $Q=\frac{(\underline{c}\times\underline{d}.\underline{a})}{(\underline{a}\times\underline{b}.\underline{c})}$ $R=\frac{(\underline{a}\times\underline{b}.\underline{d})}{(\underline{a}\times\underline{b}.\underline{c})}$

Then,

$$\underline{d}=P\underline{a}-Q\underline{b}+R\underline{c}$$

(ii) Position vector $\underline{R}(t)=\sin t\underline{i}+t\underline{i}+\cos t\underline{k}$

Let the velocity vector v(t) and acceleration vector a(t)

$\underline{v}(t)=\frac{d}{dt}\underline{R}(t)$

$=\frac{d}{dt}(\sin t\underline{i}+t\underline{j}+\cos t\underline{k})$

$=\cos t\underline{i}+1\underline{j}+(-\sin t)\underline{k}$

$$\underline{v}(t)=\cos t\underline{i}+\underline{j}-\sin t\underline{k}$$

$$\underline{a}(t)=\frac{d}{dt}(\underline{v}(t))$$

$$\underline{a}(t)=\frac{d}{dt}\left(\cos t\underline{i}+\underline{j}-\sin t\underline{k}\right)$$

$$=-\sin t\underline{i}+\underline{0}\underline{j}-\cos t\underline{k}$$

$$a(\underline{t})=-\sin t\underline{i}-\cos t\underline{k}$$

For velocity vector and acceleration vector perpendicular to each other

$$\underline{v}(t).\underline{a}(t)=0$$

Hence, $\underline{v}(t).\underline{a}(t)=\left(\cos t\underline{i}+\underline{j}-\sin t\underline{k}\right).\left(-\sin t\underline{i}-\cos t\underline{j}\right)$

$$=-(\cos t\sin t+\sin t\cos t)$$

$$=0$$

$$\underline{v}(t).\underline{a}(t)=0$$

Any time the velocity vector and acceleration vector perpendicular to each other.

(iii) $$\frac{d}{dt}(\underline{a}\times\underline{b}\times\underline{c})=\frac{d\underline{a}}{dt}\times\underline{b}\times\underline{c}+\underline{a}\times\frac{d\underline{b}}{dt}\times\underline{c}+\underline{a}\times\underline{b}\times\frac{d\underline{c}}{dt}$$

Proof $\frac{d}{dt}(\underline{a}\times\underline{b}\times\underline{c})$

Let $\underline{a}\times\underline{b}=\underline{r}$

$$\frac{d}{dt}(a\times b\times c)=\frac{d}{dt}(\underline{r}\times\underline{c})=\frac{d\underline{r}}{dt}\times c+\underline{r}\times\frac{d\underline{c}}{dt}$$

According to Basic rule 3 . $\left[\frac{d}{dt}(\underline{a}\times\underline{b})=\underline{a}\times\frac{d\underline{b}}{dt}+\frac{d\underline{a}}{dt}\times\underline{b}\right]$

$$=\frac{d\underline{r}}{dt}\times\underline{c}+\underline{r}\times\frac{d\underline{c}}{dt}$$

$$=\frac{d}{dt}\left[(\underline{a}\times\underline{b})\right]\times c+(\underline{a}\times\underline{b})\times\frac{d\underline{c}}{dt}$$

$$=\left[\underline{a}\times\frac{d\underline{b}}{dt}+\frac{d\underline{a}}{dt}\times b\right]\times c+(\underline{a}\times\underline{b})\times\frac{d\underline{c}}{dt}$$

$$=\underline{a}\times\frac{d\underline{b}}{dt}\times\underline{c}+\frac{d\underline{a}}{dt}\times\underline{b}\times\underline{c}+\underline{a}\times\underline{b}\times\frac{d\underline{c}}{dt}$$

(iv) n, a, b constant,

$$\underline{p}=a\cos nt\underline{i}+b\sin nt\underline{j}$$

$$\frac{d\underline{p}}{dt}=a-\sin nt.n\underline{i}+b-\cos nt.n\underline{j}$$

$$\underline{p}\times\frac{d\underline{p}}{dt}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ a\cos nt & b\sin nt & 0\\ -an\sin nt & -bn\cos nt & 0\end{vmatrix}$$

$$=\underline{k}\left(abn\cos nt^2-\left(anb\sin^2 nt\right)\right)$$

$$=\underline{k}(abn)\left(\cos nt^2+\sin nt^2\right)$$

$$=\underline{a\,b\,n\,k}$$

(3) (1) We take M as the amount of radium, then rate of radium decomposition will be,

$$\frac{dN}{dt}\propto -N$$

$$\frac{dN}{dt}=-\lambda N\text{----(A)}$$

λ is called as radioactivity decay constant.

Let N_0 – is amount of radium at start

N_1 – amount of radium when goes time t

Then integrating equation (A)

$$\int\frac{dN}{dt}=\int-\lambda N dt$$

$$\int_{N_0}^{N_1}\frac{1}{N}dN=\int_0^t-\lambda\,dt$$

$$\left[\ln|N|\right]_{N_0}^{N_1}+C_1=-\lambda t+C_2$$

$$\ln\left|\frac{N_1}{N_0}\right|=-\lambda t+(C_2-C_1)$$

When $t=0,\ N_1=N_0$

Substituting values,

$$\ln\left|\frac{N_0}{N_0}\right|=C_2-C_1$$

$$C_2-C_1=0$$

$$\therefore\ln\left|\frac{N_1}{N_0}\right|=-\lambda t$$

$$\frac{N_1}{N_0}=e^{-\lambda t}$$

$$N_1=N_0e^{-\lambda t}$$

If the t is taken as number of centuries and the given values are applied

$N_0 = 100\text{mg}$

$N_t = 96\text{mg}$

$t = 1$

then

$96 = 100e^{-\lambda}$

$e^{-\lambda} = 0.96$

$\lambda = \ln .96$

$\lambda = -0.04$

Amount of radium after t centuries is

$$N = N_0 e^{-0.04t}$$

Amount of radium after 2.58 centuries is

$$N = N_0 e^{(-0.04\times 2.58)}$$

$$= 106e^{(-0.04\times 2.58)}$$

$$N = 90.003\text{mg}$$

When an half-life of radium is N = 50 mg

Time taken from

$$N = N_0 e^{-0.04t}$$

$$50 = 100e^{-0.04t}$$

$$e^{-0.04t} = \frac{1}{2}$$

$$-0.04t = \ln\left(\frac{1}{2}\right)$$

$$t = \frac{-0.693}{-0.04}$$

$$\underline{\underline{t = 17.325 \text{ centuries}}}$$

(ii) $$a\left[x\frac{dy}{dx} + 2y\right] = xy\frac{dy}{dx}$$

General solution,

$$ax\frac{dy}{dx} + a2y = xy\frac{dy}{dx}$$

$$x(a-y)\frac{dy}{dx} = -2ay$$

$$x(y-a)\frac{dy}{dx} = 2ay$$

$$\frac{y-a}{2ya}=\frac{1}{x}dx$$

$$\frac{1}{2a}\left[1-\frac{a}{y}\right]dy=\frac{1}{x}dx$$

$$\frac{1}{2a}y-\frac{1}{2}\ln y=\ln x+C$$

$$y-a\ln y=2a\ln x+C$$

$$C=y-a\ln y-2a\ln x\text{ ------(1)}$$

When y = a and x = 2a, substituting the equation (1),

$$C=a-a\ln a-2a\ln 2a$$

$$C=a\left[1-(\ln a+2\ln 2a)\right]$$

$$=a\left[1-\ln a(2a)^2\right]$$

$$=a\left[1-\ln 4a^3\right]$$

The general solution is, C value substituting eq: (1)

$$y-a\left[\ln y+\ln x^2\right]=a\left[1-\ln 4a^3\right]$$

$$y-a\ln\left(yx^2\right)=a\left(1-\ln 4a^3\right)$$

(4) (i) $(x+y)dx+(x+y-2)dy=0$

Let $x+y=v$

$$1+\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{dv}{dx}-1$$

Substitute the equation,

$$v.dx+(v-2)dy=0$$

$$v+(v-2)\frac{dy}{dx}=0$$

$$(v-2)\left[\frac{dv}{dx}-1\right]=-v$$

$$(v-2)\left[\frac{dv}{dx}-1\right]=-v$$

$$(v-2)\frac{dv}{dx}=-v+(v-2)$$

$$\frac{dv}{dx}=\frac{-2}{(v-2)}$$

$$(v-2)dv=-2dx$$

$$\frac{v^2}{2}-2v=2x+C_1$$

$$\frac{(x+y)^2}{2}-2(x+y)=2x+C_1$$

(ii) $(4x+xy^2)dx+(y+x^2y)dy=0$

$$\frac{dy}{dx}=\frac{-(4x+xy^2)}{(y+x^2y)}$$

$$\frac{dy}{dx}=\frac{-x(4+y^2)}{y(1+x^2)}$$

$$\left(\frac{y}{4+y^2}\right)dy=-\frac{x}{(1+x^2)}dx$$

$$\frac{1}{2}\ln|4+y^2|=-\frac{1}{2}\ln|1+x^2|+\frac{1}{2}\ln C$$

$$\ln|4+y^2|+\ln(1+x^2)=\ln C$$

$$(4+y^2)(1+x^2)=C$$

(iii) $xy(xdy+ydx)=6y^3dy$

$$x^2ydy+xy^2dx=6y^3dy$$

$$(x^2y-6y^3)dy+xy^2dx=0$$

$$(x^2y-6y^3)\frac{dy}{dx}+xy^2=0$$

$$y(x^2-6y^2)\frac{dy}{dx}=-xy^2\ (y\neq 0)$$

$$(x^2-6y^2)\frac{dy}{dx}=-xy$$

$$\frac{dy}{dx}=\frac{-xy}{(x^2-6y^2)}$$

Let $y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$$v+x\frac{dv}{dx}=-\frac{x(vx)}{(x^2-6v^2x^2)}$$

$$v+x\frac{dv}{dx}=\frac{-v}{1-6v^2}(x\neq 0)$$

$$x\frac{dv}{dx}=\frac{-v}{1-6v^2}-v$$

$$x\frac{dv}{dx}=\frac{-v-v\left(1-6v^2\right)}{1-6v^2}$$

$$x\frac{dv}{dx}=\frac{6v^3-2v}{1-6v^2}$$

$$\frac{1-6v^2}{v(6v-2)}dv=\frac{1}{x}dx$$

$$\frac{6v^2-1}{\left(2v-6v^3\right)}dv=\frac{1}{x}dx$$

$$\left[\frac{6v^2}{2v\left(1-3v^2\right)}-\frac{1}{2v\left(1-3v^2\right)}\right]dv=\frac{1}{x}dx+C$$

$$\left\{\frac{3v}{1-3v^2}-\left[\frac{1}{2v}+\frac{\frac{3}{2}v}{\left(1-3v^2\right)}\right]\right\}dv=\frac{1}{x}dx+C$$

$$\frac{3v}{1-3v^2}dv-\frac{1}{2v}dv-\frac{3v}{2\left(1-3v^2\right)}dv=\frac{1}{x}dx+C$$

$$-\frac{1}{2}\ln\left|1-3v^2\right|-\frac{1}{2}\ln|v|+\frac{1}{4}\ln\left|1-3v^2\right|=\ln x+C$$

$$-2\ln\left|1-3v^2\right|-2\ln|v|+\ln\left|1-3v^2\right|=\ln x+C$$

$$\ln\left|v^2\right|-\ln\left|1-3v^2\right|=-\ln x+C$$

$$\ln\left|\frac{v^2x}{\left(1-3v^2\right)}\right|=C$$

(iv) $xydy=(y+1)(1-x)dx$

$$xy\frac{dy}{dx}=(y+1)(1-x)$$

$$\frac{y}{(y+1)}dy=\frac{(1-x)}{x}dx+C_1$$

$$\left(1-\frac{1}{y+1}\right)dy=\left(\frac{1}{x}-1\right)dx+C$$

$$y-\ln(y+1)=\ln x-x+C_1$$

$$y=\ln(y+1)+\ln(x)-x+C_1$$

$$y=\ln x(y+1)-x+C_1$$

(v) $\left(x^2+y^2\right)dx+2xydy=0$

$$2xy\frac{dy}{dx}=-\left(x^2+y^2\right)$$

Let $y=vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

Then substituting,

$$v+x\frac{dv}{dx}=\frac{-\left(x^2+(vx)^2\right)}{2x(vx)}$$

$$v+x\frac{dv}{dx}=\frac{-\left(1-v^2\right)}{2v}$$

$$v+x\frac{dv}{dx}=\frac{-\left(1+v^2\right)}{2v}-v$$

$$x\frac{dv}{dx}=\frac{-\left(1+v^2\right)-2v^2}{2v}$$

$$x\frac{dv}{dx}=\frac{-\left(3v^2+1\right)}{2v}$$

$$\frac{2v}{-\left(3v^2+1\right)}=\frac{1}{x}dx$$

$$\frac{1}{-3}\ln\left|3v^2+1\right|=\ln|x|+C$$

$$\ln|x|+\frac{1}{3}\ln\left|1+3\left(\frac{y}{x}\right)^2\right|+C_1=0$$

$$\ln|x|+\ln\left|1+\frac{3y^2}{x^2}\right|^{\frac{1}{3}}+C_1=0$$

$$\ln\left|x\left(1+\frac{3y^2}{x^2}\right)^{\frac{1}{3}}\right|=C$$

(5) (i) $(\cos y+y\cos t)dt+(\sin t-t\sin y)dy=0$

$$(\sin t-t\sin y)\frac{dy}{dx}+(\cos y+y\cos t)=0$$

Let $f(y,t)=\sin t-t\sin y$ and $g(y,t)=\cos y+y\cos t$

Integrating, f(y, t)

$$\frac{\partial f}{\partial t} = \cos t - \sin y$$

Integrating g(y, t)

$$\frac{\partial y}{\partial t} = -\sin y + \cos t$$

Hence

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial t}$$

Therefore differential equation is exact
Then U(y, t) = C is a solution

Then, $\frac{\partial u}{\partial t} = g(y,t) = \cos y + y\cos t$

$$U = \int (\cos y + y\cos t)\,dt + f(y)$$

$$U = t\cos y + y\sin t + f(y)$$

Then, $\frac{\partial u}{\partial y} = t(-\sin y) + \sin t + f'(y)$ ------(A)

Then comparing $\frac{\partial u}{\partial y}$ and $f(y,t)$

$$\therefore f'(y) = 0$$

Therefore solution is,

$$C = t\cos y + y\sin t$$

(ii) $(3x^2 + y^2)\,dx - 2xy\,dy = 0$

$$2xy\frac{dy}{dx} - (3x^2 + y^2) = 0 \text{ ----(I)}$$

Let, $g(x,y) = 2xy \quad f(x,y) = -(3x^2 + y^2)$

$$\frac{\partial g}{\partial x} = 2y \qquad \frac{\partial f}{\partial y} = -2y$$

$\frac{\partial g}{\partial x} \neq \frac{\partial f}{\partial y}$ Therefore this equation is not exact.

Then use integrating factor method,

Integrating factor $P = \frac{\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)}{g(x,y)} = \frac{-2y - 2y}{2xy} = \frac{-2}{x}$

$$P = e^{\int \frac{-2}{x} dx} = e^{-2\ln x} = e^{\ln(x)^{-2}} = e^{\ln \frac{1}{x^2}} = \frac{1}{x^2}$$

Integrating factor $= \frac{1}{x^2}$

Multiplying the equation (1)

$$\frac{1}{x^2} \times 2xy \frac{dy}{dx} - \frac{1}{x^2}\left(3x^2 + y^2\right) = 0$$

$$\frac{2y}{x}\frac{dy}{dx} - \left(3 + \frac{y^2}{x^2}\right) = 0 \quad \text{--------- (2)_}$$

Let the solution is $u = c$

$$\text{Eq: (2)} \Rightarrow \frac{\partial u}{\partial x} = \int\left(-3 - \frac{y^2}{x^2}\right)dx + f(y)$$

$$u = -3x - y^2\left(-\frac{1}{x}\right) + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{2y}{x} + f'(y) \text{-------(3)}$$

Comparing eq. (2) and (3) find f'(y)

$$\frac{2y}{x} + f'(y) = \frac{2y}{x}$$

$$f'(y) = 0$$

Therefore solution is $U = \frac{y^2}{x} - 3x$

(iii) $\frac{xy-1}{x^2y}dx - \frac{1}{xy^2}dy = 0$

$$\frac{1}{xy^2}\frac{dy}{dx} - \frac{(xy-1)}{x^2y} = 0$$

Let $g(x,y) = \frac{1}{xy^2}$ $f(x,y) = \frac{-(xy-1)}{x^2y}$

$$\frac{\partial g}{\partial x} = \frac{-1}{x^2y^2} \qquad \frac{\partial f}{\partial y} = \frac{-1}{x^2y^2}$$

Therefore

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$

Then, let the solution is $U_{(x,y)} = C$

$$\frac{\partial u}{\partial x} = f(x,y) = \frac{1-xy}{x^2y}$$

$$U = \frac{1}{y}\int\frac{1}{x^2}dx - \int\frac{1}{x}dx + f(y)$$

$$U = \frac{1}{yx} - \ln x + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{-1}{xy^2} + f'(y)$$

Comparing the above equations and g(x, y)

$$\frac{-1}{x^2y} + f'(y) = -\frac{1}{x^2y}$$

$$\therefore f'(y) = 0$$

Therefore solution is, $C = \frac{1}{yx} - \ln x$

(iii) $\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

$$\left(2xye^{xy^2} - 3y^2\right)\frac{dy}{dx} + \left(y^2e^{xy^2} + 4x^3\right) = 0$$

Let, $g = \left(2xye^{xy^2} - 3y^2\right)$, $f = \left(y^2e^{xy^2} + 4x^3\right)$

$$\frac{\partial g}{\partial x} = 2y\left(e^{xy^2} + xe^{xy^2}y^2\right)$$

$$= 2xy^3e^{xy^2} + 2ye^{xy^2}$$

$$\frac{\partial f}{\partial y} = 2ye^{xy^2} + y^2e^{xy^2}x2y$$

$$= 2ye^{xy^2} + 2y^3xe^{xy^2}$$

$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$ Therefore the differential equation is exact.

Let the solution is $U(x, y) = C$

$$\frac{\partial u}{\partial x} = f(x,y) = y^2e^{xy^2} + 4x^3$$

$$U = \int y^2e^{xy^2}dx + 4\int x^3dx + f(y)$$

$$= \cancel{y^2}\frac{e^{xy^2}}{\cancel{y^2}} + \cancel{4}\frac{x^4}{\cancel{4}} + f(y)$$

$$= e^{xy^2} + x^4 + f(y)$$

$$U = e^{xy^2} + x^4 + f(y)$$

$$\frac{\partial u}{\partial y} = e^{xy^2} 2yx + f'(y)$$

Comparing with g(x, y)

$$\frac{\partial u}{\partial y} = 2xye^{xy^2} + f'(y) = 2xye^{xy^2} - 3y^2$$

$$f'(y) = -3y^2$$

$$f(y) = -3\frac{y^3}{3} = -y^3$$

$\therefore$ Solution is

$$U = C = e^{xy^2} + x^4 + f(y)$$

$$C = e^{xy^2} + x^4 - y^3$$

(v) $(x+y)dx - (x-y)dy = 0$

$$(x-y)\frac{dy}{dx} - (x+y) = 0$$

$$g(x,y) = (x-y) \qquad f(x,y) = -(x+y)$$

$$\frac{\partial g}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = -1$$

This equation is not exact

$$\therefore \frac{dy}{dx} = \frac{x+y}{x-y} ----(1)$$

To solve this differential equation,

Use this substituent $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Substitute (1)

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx}$$

$$x\frac{dv}{dx} = \frac{1+v}{1-v} - v$$

$$x\frac{dv}{dx}=\frac{1+\cancel{v}-\cancel{v}+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}dv=\frac{1}{x}dx$$

$$\frac{1}{1+v^2}dv-\frac{v}{1+v^2}dv=\frac{1}{x}dx$$

$$\tan^{-1}v-\frac{1}{2}\ln\left|1+v^2\right|=\ln x+C$$

$$\tan^{-1}v-\ln\left[\left(1+v^2\right)^{1/2}x\right]=C$$

Then substitute, $v=\frac{y}{x}$

$$\underline{\tan^{-1}\left(\frac{y}{x}\right)-\ln\left|\left(1+\frac{y^2}{x^2}\right)^{1/2}x\right|=C}$$

(vi) This is a printing mistaken problem.