MPZ 3230 - Assignment # 02
Academic Year – 2007

(1) The expression for the electric charge E in a particular circuit is given by the differential equation

$$\frac{d^2E}{dt^2}+2k\frac{dE}{dt}+4E=\cos t$$

Where k is a constant.

(a) Find the complementary function in each of the three cases,

$$k=\frac{5}{2},\ k=1 \text{ and } k=2$$

(b) In the case k = 2, obtain the general solution.

(c) In the case k = 1, solve the differential equation completely given that $E=0$ and $\frac{dE}{dt}=1$ when $t=0$.

(2) (a) Obtain the general solution of $\frac{d^3y}{dx^3}+4\frac{d^2y}{dx^2}+4\frac{dy}{dx}=8e^{-2x}$

(b) Obtain the general solution of $\frac{du}{dt}=u-4v,\ \frac{dv}{dt}=5u+5v$

(c) Using variation of parameter Method, obtain the general solution of

$$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+3y=\left(1+e^{-x}\right)^{-1}$$

(d) Using Trial Functions method, obtain the general solution of

$$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$$

(3) (i) Using the definition of the Laplace transform, find the Laplace transform of $f(x)=x^3$ and $f(x)=xe^{-x}$

Hence deduce the Laplace transform of $\left(1+xe^{-x}\right)^3$

(ii) Find the inverse Laplace transforms of the

(a) $\frac{2s+7}{3s^2+5}$ (b) $\frac{1}{4s^2-8s}$ (c) $\frac{s+3}{s^2+2s+5}$

(iii) (a) Solve the following initial value problem $y''-4y'+3y=e^{3x}$ by using the Laplace transform method, given that $y(0)=1$ and $y'(0)=2$

(b) Solve $y''+9y=\cos 2t$ if $y(0)=1$, $y\left(\pi/2\right)=-1$ by using the Laplace transform method.

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(4) (i) Let $A=\begin{pmatrix}1 & 2 & P\\ 2 & 3 & -2\\ P & 1 & -P\end{pmatrix}$, $B=\begin{pmatrix}1 & 0 & -4\\ 0 & -1 & 2\\ 2 & -1 & -2\end{pmatrix}$

(a) Find the values of P for which the matrix A is singular.

(b) If P = 0 and I is an unit matrix, prove that BA + A = 2I

(c) Using the expression BA + A = 2I obtain an expression for A^{-1} and deduce the A^{-1}.

(ii) Find Lower and upper triangular matrices L and U such that LU is the matrix $A=\begin{pmatrix}1 & 1 & 2 & 1\\ 0 & 1 & 1 & 1\\ 1 & 2 & 1 & 1\\ 2 & 2 & 4 & 0\end{pmatrix}$

(5) (i) If A and B are n x n matrices, then show that adj(AB) = adj B . adj A

(ii) Find the inverse of matrix

$\begin{pmatrix}3 & -3 & 4\\ 2 & -3 & 4\\ 0 & -1 & 1\end{pmatrix}$ By using the method of elementary row transformations.

(iii) Without expanding show that,

$$\begin{vmatrix}a & -b & -a & b\\ b & a & -b & -a\\ c & -d & c & -d\\ d & c & d & c\end{vmatrix}=4(a^2+b^2)(c^2+d^2)$$

(6) (i) Find the value of $\begin{vmatrix}1^2 & 2^2 & 3^2 & 4^2\\ 2^2 & 3^2 & 4^2 & 5^2\\ 3^2 & 4^2 & 5^2 & 6^2\\ 4^2 & 5^2 & 6^2 & 7^2\end{vmatrix}$.

(ii) If $z=e^{xy^2}$, $x=t\cos t$, $y=t\sin t$, compute $\frac{dz}{dt}$ at $t=\frac{\pi}{2}$

(iii) A function f(x, y) is given as, $u=\frac{x^2y^2}{x^2+y^2}$

Prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$

(iv) Prove De-Morgan's theorems using logical expressions.

Please send your assignment **on or before due date according to activity diary** to the following address.**Please send your answer with your address (write back of your answer sheet) and when you are doing assignment use both sides of paper.**

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Model answer – Assignment No. 02
MPZ 3230 – Academic Year 2007

(1) $\dfrac{d^2E}{dt^2}+2k\dfrac{dE}{dt}+4E=\cos t$

(i) When $k=\dfrac{5}{2}$

Equation, $\dfrac{d^2E}{dt^2}+5\dfrac{dE}{dt}+4E=\cos t$

The homogeneous form is, $\dfrac{d^2E}{dt^2}+5\dfrac{dE}{dt}+4E=0$

The characteristic equation is,

$$m^2+5m+4=0$$
$$(m+4)(m+1)=0$$

Let m_1 and m_2 are the roots of the characteristic equation:

$$\therefore m_1=-4, m_2=-1$$

Therefore, the complementary function is, E_c

$E_c=Ae^{-4t}+Be^{-t}$ where A and B are arbitrary constant.

(ii) When $k=1$

Equation, $\dfrac{d^2E}{dt^2}+2\dfrac{dE}{dt}+4E=\cos t$

The homogeneous form is, $\dfrac{d^2E}{dt^2}+2\dfrac{dE}{dt}+4E=0$

Characteristic equations is, $m^2+2m+4=0$

$$(m+1)^2+3=0$$
$$(m+1)^2=-3$$
$$(m+1)^2=3i^2$$
$$m+1=\pm\sqrt{3}\,i$$

Let m_1 and m_2 are the roots of the characteristic equation.

$$\therefore m_1=-1+\sqrt{3}\,i \qquad m_2=-1-\sqrt{3}\,i$$

Therefore, the complementary function is,

$E_c=e^{-t}(A\cos\sqrt{3}t+B\sin\sqrt{3}t)$ Where A and B are arbitrary constant.

(ii) When $k = 2$

Equation, $\frac{d^2E}{dt^2} + 4\frac{dE}{dt} + 4E = \cos t$

The homogeneous form is, $\frac{d^2E}{dt^2} + 4\frac{dE}{dt} + 4E = 0$

The characteristic equation is,

$$m^2 + 4m + 4 = 0$$
$$(m+2)^2 = 0$$
$$m = -2$$

Let m_1 and m_2 are the roots of the characteristic equation,

$$\therefore m_1 = -2, m_2 = -2$$

Complementary function is, $E_c = Ae^{-2t} + Bte^{-2t}$ Where A and B are arbitrary constant

(b) When $k = 2$

$$\frac{d^2E}{dt^2} + 4\frac{dE}{dt} + 4E = \cos t$$

We have found out that the complementary function,

$$E_c = Ae^{-2t} + Bte^{-2t}$$

To find the particular integral we will assume the trial function of the form is,

$E_p = \alpha \cos t + \beta \sin t$, Where α and β are unknown constants.

$$\frac{dE_p}{dt} = \alpha(-\sin t) + \beta(\cos t)$$

$$\frac{d^2E_p}{dt^2} = \alpha(-\cos t) + \beta(-\sin t)$$

Substituting in the equation,

$$\frac{d^2E_p}{dt^2} + 4\frac{dE_p}{dt} + 4E_p = \cos t$$

$$\alpha(-\cos t) + \beta(-\sin t) + 4\left[\alpha(-\sin t) + \beta\cos t\right] + 4(\alpha\cos t + \beta\sin t) = \cos t$$
$$(-\alpha + 4\beta + 4\alpha)\cos t + (-\beta - 4\alpha + 4\beta)\sin t = \cos t$$

$-\alpha + 4\beta + 4\alpha = 1$ ---------- (1)
$-\beta - 4\alpha + 4\beta = 0$ ---------- (2)

$(1)\times 4+(2)\times 3 \Rightarrow$

$4(3\alpha+4\beta)+3(-4\alpha+3\beta)=4$

$16\beta+9\beta=4$

$25\beta=4$

$\beta=\frac{4}{25}$

$4\alpha=3.\frac{4}{25}$

$\alpha=\frac{3}{25}$

$\therefore$ Particular intrigal $E_p=\frac{3}{25}\cos t+\frac{4}{25}\sin t$

$=\frac{(3\cos t+4\sin t)}{25}$

$\therefore$ The general solution is $E=E_c+E_p$

$$E=Ae^{-2t}+Bte^{-2t}+\frac{(3\cos t+4\sin t)}{25}$$

(c) When $k=1$,

$$\frac{d^2E}{dt^2}+2\frac{dE}{dt}+4E=\cos t$$

The complementary function is given by part (a),

$$E_c=e^{-t}(A\cos\sqrt{3}t+B\sin\sqrt{3}t)$$

We assume the trial function of the form is,

$E_p=\theta\cos t+\gamma\sin t$, Where θ and γ are unknown functions.

$E_p'=\theta(-\sin t)+\gamma\cos t$

$E_p''=\theta(-\cos t)+\gamma(-\sin t)$

Substituting in the equation,

$\frac{d^2E}{dt^2}+2\frac{dE}{dt}+4E=\cos t$

$(-\theta\cos t-\gamma\sin t)+2(-\theta\sin t+\gamma\cos t)+4(\theta\cos t+\gamma\sin t)=\cos t$

$(-\theta+2\gamma+4\theta)\cos t+(-\gamma-2\theta+4\gamma)\sin t=\cos t$

By equating,

$3\theta+2\lambda=1$ ------- (1)

$$3\gamma - 2\theta = 0 \quad \text{------ (2)}$$

$(1)\times 2 + (2)\times 3 \Rightarrow 4\gamma + 9\gamma = 2 \qquad \theta = \frac{3}{13}$

$$\gamma = \frac{2}{13}$$

$$E_p = \frac{3}{13}\cos t + \frac{2}{13}\sin t$$

$$E_p = \frac{(3\cos t + 2\sin t)}{13}$$

$\therefore$ General solution is, $E = E_C + E_p$

$$E = e^{-t}(A\cos\sqrt{3}t + B\sin\sqrt{3}t) + \frac{(3\cos t + 2\sin t)}{13}$$

When applying bounding conditions,

$E = 0$ when $t = 0$

Substituting to general solution,

$$0 = A + 0 + \frac{3}{13} + 0$$

$$A = -\frac{3}{13}$$

$$E = e^{-t}\left(A\cos\sqrt{3}t + B\sin\sqrt{3}t\right) + \frac{1}{13}(3\cos t + 2\sin t)$$

$$\frac{dE}{dt} = e^{-t}\left[A\left(-\sin\sqrt{3}t\right)\right]\sqrt{3} + \left(B\cos\sqrt{3}\right) + \left(A\cos\sqrt{3}t + B\sin\sqrt{3}t\right)(-1)e^{-t} + \frac{1}{\sqrt{3}}(-3\sin t + 2\cos t)$$

$$\frac{dE}{dt} = e^{-t}\left[\left(\sqrt{3}B - A\right)\cos\sqrt{3}t - \left(B + \sqrt{3}A\right)\sin\sqrt{3}t\right] + \frac{1}{13}(2\cos t - 3\sin t)$$

$\frac{dE}{dt} = 1$. When $t = 0$. Substituting to this equation

$$1 = 1\left[\left(\sqrt{3}B - A\right) - 0\right] + \frac{1}{13}(2\times 1 - 0)$$

$$1 = \sqrt{3}B - A + \frac{2}{13}$$

$$\sqrt{3}B = \frac{8}{13}$$

$$B = \frac{8}{13\sqrt{3}}$$

The general solution is,

$$E = e^{-t}\left(-\frac{3}{13}\cos\sqrt{3}t + \frac{8}{13\sqrt{3}}\sin\sqrt{3}t\right) + \frac{1}{13}(3\cos t + 2\sin t)$$

$$E = \frac{e^{-t}}{13}\left[\frac{8}{\sqrt{3}}\sin\sqrt{3}t - 3\cos\sqrt{3}t\right] + \frac{1}{13}(3\cos t + 2\sin t)$$

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(2) (a) $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 8e^{-2x}$

The homogeneous form is, $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 0$

Characteristic equation is $m^3 + 4m^2 + 4m = 0$

$m(m^2 + 4m + 4) = 0$

$m(m+2)^2 = 0$

Let m_1, m_2 and m_3 are the roots of the equation.

$m_1 = 0,\ m_2 = -2,\ m_3 = -2$

$\therefore$ The complementary function is,

$y_c = Ae^{0x} + Be^{-2x} + Cxe^{-2x}$ Where A, B, C are arbitrary constant.

$\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} = 8e^{-2x}$

To find the particular interregnal, assume the trial function of the form is,

$y_p = \alpha e^{-2x}$

$y_p' = (-2)\alpha e^{-2x}$

$y_p'' = 4\alpha e^{-2x}$

$y_p''' = (-8)\alpha e^{-2x}$

Substituting value for the equation,

$[-8\alpha]e^{-2x} + 4[4\alpha]e^{-2x} + 4[-2\alpha]e^{-2x} = 8e^{-2x}$

$-8\alpha e^{-2x} + 16\alpha e^{-2x} - 8\alpha e^{-2x} = 8\alpha e^{-2x}$

Particular Interregnal is $= 0$

$\therefore$ General solution is, $y = A + Be^{-2x} + Cxe^{-2x}$

(b) $\frac{du}{dt} = u - 4v$ $\frac{dv}{dt} = 5u + 5v$

Let $u = \alpha e^{\mu t}$ and $v = \beta e^{\mu t}$

$\frac{du}{dt} = \mu\alpha e^{\mu t}$ $\frac{dv}{dt} = \mu\beta e^{\mu t}$

Substituting in the above equations, we obtain algebraic equation,

$\mu\alpha e^{\mu t} = \alpha e^{\mu t} - 4\left(\beta e^{\mu t}\right)$

$(\mu - 1)\alpha + 4\beta = 0$ ---------- (1)

$(\mu - 5)\beta - 5\alpha = 0$ ---------- (2)

$$\begin{vmatrix} \mu - 1 & 4 \\ -5 & \mu - 5 \end{vmatrix} = 0$$

This gives us,

$(\mu - 1)(\mu - 5) + 20 = 0$

$\mu^2 - \mu - 5\mu + 5 + 20 = 0$

$\mu^2 - \mu - 5\mu + 5 + 20 = 0$

$\mu^2 - 6\mu + 25 = 0$

$(\mu - 3)^2 + 16 = 0$

$(\mu - 3)^2 = 16i^2$

Since $\mu = 3 \pm 4i$ are roots

Let $\mu = 3 + 4i$,

Then $u = \alpha e^{(3+4i)t}$, $v = \beta e^{(3+4i)t}$

The characteristic roots are,

Let $u = \alpha_1 e^{(3+4i)t}$ $v = \beta_1 e^{(3+4i)t}$

$\therefore \left[1 - (3 + 4i)\right]\alpha - 4\beta_1 = 0$

$(-2 - 4i)\alpha_1 - 4\beta_1 = 0$

$-(1 + 2i)\alpha_1 - 2\beta_1 = 0$

Let $\alpha_1 = 2A$ $\therefore$ $\beta_1 = -(1 + 2i)A$

Let $\mu = 3 - 4i$

Then $u = \alpha_2 e^{(3-4i)t}$ $v = \beta_2 e^{(3-4i)t}$

$[1-(3-4i)]\alpha_2 - 4\beta_2 = 0$

$(-2+4i)\alpha_2 - 4\beta_2 = 0$

$\alpha_2 = 2B \quad \therefore \quad \beta_2 = -(1-2i)B$

$\therefore$ The general solution is,

$$u = 2Ae^{(3+4i)t} + 2Be^{(3-4i)t}$$
$$v = -(1+2i)Ae^{(3+4i)t} - (1-2i)Be^{(3-4i)t}$$

For Arbitrary A and B

$$u = 2e^{3t}\left[(A+B)(\cos 4t) + (A-B)i\sin 4t\right]$$
$$v = -e^{3t}\left[(A+B)+2i(A-B)\right]\cos 4t + \left[-2(A+B)+i(A-B)\sin 4t\right]$$

Let $C = A + B$ and $D = i(A - B)$

$$\therefore u = 2e^{3t}(C\cos 4t + D\sin 4t)$$
$$v = -e^{3t}\left[(C+2D)\cos 4t + (-2C+D)\sin 4t\right]$$

C and D are arbitrary constant.

(c) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = \left(1+e^{-x}\right)^{-1}$

Homogeneous form is,

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

Characteristic equation is,

$$m^2 - 4m + 3 = 0$$
$$(m-3)(m-1) = 0$$

$\therefore$ Let, m_1, m_2 are the roots of the equation

$$m_1 = 3 \quad m_2 = 1$$

$\therefore$ The complementary function is,

$y_c = Ae^{3x} + Be^{x}$, where A and B are arbitrary constant,

$y_1 = e^{3x}, \quad y_2 = e^{x}$

Then the un-homogeneous equation, find particular integral,

The wranstian , $w(x) = y_1'y_2 - y_1y_2'$

$$w(x) = y_1'y_2 - y_1y_2'$$
$$= 3e^{3x}e^{x} - e^{3x}e^{x}$$
$$= 3e^{4x} - e^{4x}$$

$$= 2e^{4x}$$

$$\int\frac{y_2 f(x)\,dx}{w(x)} = \int\frac{e^{x}\left(1+e^{-x}\right)^{-1}}{2e^{4x}}dx = \frac{1}{2}\int\frac{1}{e^{3x}\left(1+e^{-x}\right)}dx$$

$$= \frac{1}{2}\int\frac{e^{-3x}}{\left(1+e^{-x}\right)}dx$$

Apply, $\begin{bmatrix} 1+e^{-x}=u \\ (-e^{-x})=\dfrac{du}{dx} \\ dx=\dfrac{1}{-e^{-x}}du=-e^{x}du \end{bmatrix}$

$$= -\frac{1}{2}\int\frac{e^{-3x}}{u}\left(-e^{x}\right)du$$

$$= -\frac{1}{2}\int\frac{e^{-2x}}{u}du = -\frac{1}{2}\int\frac{(e^{-x})(e^{-x})}{u}du$$

$$= -\frac{1}{2}\int\frac{(u-1)^2}{u}du$$

$$= -\frac{1}{2}\int\frac{u^2-2u+1}{u}du$$

$$= -\frac{1}{2}\int u.du + \int du - \frac{1}{2}\int\frac{1}{u}du$$

$$= -\frac{1}{2}\frac{u^2}{2} + u - \frac{1}{2}\ln u$$

$$\int\frac{y_2 f(x)\,dx}{wx} = \frac{1}{4}\left(1+e^{-x}\right)^2 + \left(1+e^{-x}\right) - \frac{1}{2}\ln\left(1+e^{-x}\right) \text{ --------- (1)}$$

$$\int\frac{y_1 f(x)}{wx}dx = \int\frac{e^{3x}\left(1+e^{-x}\right)^{-1}}{2e^{4x}}dx = \int\frac{e^{-x}}{2\left(1+e^{-x}\right)}dx$$

$$= \frac{1}{2}\int\frac{e^{-x}}{\left(1+e^{-x}\right)}dx$$

$$= \frac{1}{2}\ln\left|\left(1+e^{-x}\right)\right|$$

$$y_p = \frac{y_1(x)}{1}\int\frac{y_2 f(x)}{w(x)}dx - y_2(x)\int\frac{y_1 f(x)}{w(x)}dx$$

$$= e^{3x}\left[\frac{1}{4}\left(1+e^{-x}\right)^2 + \left(1+e^{-x}\right) - \frac{1}{2}\ln\left(1+e^{-x}\right)\right] - e^{x}\frac{1}{2}\ln\left(1+e^{-x}\right)$$

$$y_p = \frac{e^{3x}}{4}\left(1+e^{-x}\right)^2 - \frac{1}{2}\ln\left(1+e^{-x}\right)\left(e^{3x}+e^{x}\right) + e^{3x}\left(1+e^{-x}\right)$$

Complementary function is $E_c = Ae^x + Be^{3x}$

General solution is $E = E_c + E_p$

$$E = Ae^x + Be^{3x} - \frac{1}{2}\left[\left(e^{3x}+e^x\right)\ln\left(1+e^{-x}\right)\right] + \left(1+e^{-x}\right)e^{3x} + \frac{1}{4}e^{3x}\left(1+e^{-x}\right)^2$$

(d) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}x^2$

The Homogeneous function is,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

Characteristic equation is,

$$m^2 - 6m + 9 = 0$$
$$(m-3)^2 = 0$$
$$m = 3$$

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Complementary function is,

$y_c = Ae^{3x} + Bxe^{3x}$ (A & B are arbitrary constant)

We will assume the trial function is,

$$y_p = \left(\alpha_0 x^2 + \alpha_1 x + \alpha_2\right)e^{3x}$$
$$y_p' = \left(3\alpha_0 x^2 + 3\alpha_1 x + 3\alpha_2 + 2\alpha_0 x + \alpha_1\right)e^{3x}$$
$$y_p'' = \left(9\alpha_0 x^2 + 9\alpha_1 x + 9\alpha_2 + 6\alpha_0 x + 3\alpha_1 + 6\alpha_0 x + 3\alpha_1 + 2\alpha_0\right)e^{3x}$$

Substituting in the equation,

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}x^2$$

$$y'' - 6y' + 9y = e^{3x}x^2$$

$$\left(9\alpha_0 x^2 + (9\alpha_1 + 12\alpha_0)x + 6\alpha_1 + 2\alpha_0 + 9\alpha_2\right)e^{3x} - 6(3\alpha_0 x^2 + (3\alpha_1 + 2\alpha_0)x + 3\alpha_2 + \alpha_1)e^{3x}$$
$$+9(\alpha_0 x^2 + \alpha_1 x + \alpha_2)e^{3x} = e^{3x}x^2$$

$$e^{3x}[x^2(9\alpha_0 - 18\alpha_0 + 9\alpha_0) + (18\alpha_1 - 18\alpha_1 + 12\alpha_0 - 12\alpha_0)$$
$$+6\alpha_1 - 6\alpha_1 + 2\alpha_0 + 18\alpha_2 - 18\alpha_2] = 0$$

$\alpha_0 = 0, \alpha_1 = 0, \ \alpha_2 = 0$

We can find above results by equating respective coefficients,

$$y_p = 0$$

$\therefore$ General solution is $y = Ae^{3x} + Bxe^{3x}$

(3) (i) $f(x) = x^3$

$$= L\left[x^3\right]$$

$$= \int_0^\infty x^3 e^{-sx} dx$$

$$= \int_0^\infty x^3 \frac{d}{dx}\left[\frac{e^{-sx}}{-s}\right] dx$$

$$= \left[x^3 \frac{e^{-sx}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-sx}}{-s} 3x^2 dx$$

$$= \frac{3}{s}\int_0^\infty x^2 \frac{d}{dx}\left[\frac{e^{-sx}}{-s}\right] dx$$

$$= \frac{3}{s}\left[x^2 . \frac{e^{-sx}}{-s}\right]_0^\infty - \frac{3}{s}\int_0^\infty \frac{e^{-sx}}{-s} .2x \ dx$$

$$= \frac{6}{s^2}\int_0^\infty x . \frac{d}{dx}\left[\frac{e^{-sx}}{-s}\right] dx$$

$$= \frac{6}{s^2}\left[\frac{e^{-sx}}{-s} x\right]_0^\infty - \frac{6}{s^2}\int_0^\infty \frac{e^{-sx}}{-s} .1.dx$$

$$= \frac{6}{s^3}\int_0^\infty e^{-sx} dx$$

$$= \frac{6}{s^4}$$

(ii) $f(x) = xe^{-x}$

$$L\left[f(x)\right] = \int_0^\infty xe^{-x} e^{-sx} dx$$

$$= \int_0^\infty xe^{-x(s+1)} dx$$

$$= \int_0^{\infty} x \frac{d}{dx}\left[\frac{e^{-(s+1)x}}{-(s+1)}\right]dx$$

$$= \left[x.\frac{e^{-(s+1)x}}{-(s+1)}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-(s+1)x}}{-(s+1)}.dx$$

$$= \frac{1}{(s+1)} \int_0^{\infty} e^{-(s+1)x} dx$$

$$= \frac{1}{s+1}\left[\frac{e^{-(s+1)x}}{-(s+1)}\right]_0^{\infty}$$

$$= \frac{1}{(s+1)^2}$$

(iii) $f(x) = (1 + xe^{-x})^3$

Given, $L\{x^3\} = \frac{6}{s^4}$, $L\{xe^{-x}\} = \frac{1}{(s+1)^2}$

$$L\left\{\left(1 + xe^{-x}\right)^3\right\}$$

Let $1 + xe^{-x} = y$

$$L\{y^3\} = \frac{3}{s} L\{y^2\}$$

$$= \frac{3.2}{s^2} L\{y\}$$

$$= \frac{6}{s^2} L\{1 + xe^{-x}\}$$

$$= \frac{6}{s^2}\{L\{1\} + L\{xe^{-x}\}\}$$

$$= \frac{6}{s^2}\left\{\frac{1}{s} + \frac{1}{(1+s)^2}\right\}$$

$$= \frac{6}{s^3} + \frac{6}{s^2(1+s)^2}$$

(ii) Inverse Laplace

(a) $F(s) = \frac{2s+7}{3s^2+5}$

$$= 2\left(\frac{s}{3s^2+5}\right)+7\left(\frac{1}{3s^2+5}\right)$$

$$= \frac{2}{3}\left(\frac{s}{s^2+5/3}\right)+\frac{7}{3}\left(\frac{1}{s^2+5/3}\right)$$

$$= \frac{2}{3}\left(\frac{s}{s^2+\left(\sqrt{5/3}\right)^2}\right)+\frac{7}{\sqrt{3}\sqrt{5}}\left(\frac{\left(\sqrt{5/3}\right)}{s^2+\left(\sqrt{5/3}\right)^2}\right)$$

$$= \underline{\underline{\frac{2}{3}\cos\left(\sqrt{\frac{5}{3}}\right)t+\frac{7}{\sqrt{15}}\sin\left(\sqrt{\frac{5}{3}}\right)t}}$$

(b) $F(s)=\dfrac{1}{4s^2-8s}$

$$= \frac{1}{4s(s-2)}$$

$$\frac{1}{4s(s-2)}=\frac{A}{4s}+\frac{B}{s-2}$$

$$A=-\frac{1}{2}\ ,\ B=\frac{1}{8}$$

$$F(s)=-\frac{1}{8s}+\frac{1}{8(s-2)}$$

$$= -\frac{1}{8}\left(\frac{1}{s}\right)+\frac{1}{8}\left(\frac{1}{s-2}\right)$$

$$= \underline{\underline{\frac{1}{8}\left(-1+e^{2t}\right)}}$$

(c) $\dfrac{s+3}{s^2+2s+5}$

$$f(s)=\frac{s+3}{s^2+2s+5}$$

$$= \frac{s+1+2}{(s+1)^2+4}$$

$$= L^{-1}\left[\frac{s+3}{s^2+2s+5}\right]$$

$$= L^{-1}\left[\frac{(s+1)+2}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] + L^{-1}\left[\frac{2}{(s+1)^2+2^2}\right]$$
$$= e^{-t}\cos 2t + e^{-t}\sin 2t$$
$$= e^{-t}(\cos 2t + \sin 2t)$$

(iii) (a) $y'' - 4y' + 3y = e^{3x}$

Given that $y(0) = 1$ & $y'(0) = 2$

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = e^{3x}$$

Taking the Laplace transform:

$$\left[s^2y(s) - s.y(0) - y'(0) - 4\left(sy(s) - y(0)\right) + 3y(s)\right] = L\left[e^{3x}\right]$$

$$\left(s^2 - 4s + 3\right)y(s) - s - 2 + 4 = \frac{1}{s-3}$$

$$\left(s^2 - 4s + 3\right)y(s) = \frac{1}{s-3} + s - 2$$

$$\left(s^2 - 4s + 3\right)y(s) = \frac{1 + (s-2)(s-3)}{(s-3)}$$

$$y(s) = \frac{1 + s^2 - 5s + 6}{\left(s^2 - 4s + 3\right)(s-3)}$$

$$= \frac{s^2 - 5s + 7}{(s-1)(s-3)(s-3)}$$

Find, partial factorials of the equation,

$$\frac{s^2 - 5s + 7}{(s-1)(s-3)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-3)} + \frac{c}{(s-3)^2}$$

$A = ¾$, $C = ½$, $B = ¼$

$$y(s) = \frac{3}{4(s-1)} + \frac{1}{4(s-3)} + \frac{1}{2(s-3)^2}$$

$$y(x) = L^{-1}\left[y(s)\right]$$

$$= \frac{3}{4}e^{x} + \frac{1}{4}e^{3x} + \frac{1}{2}xe^{3x}$$

(b) $y'' + 9y = \cos 2t$

$$\frac{d^2y}{dx} + 9y = \cos 2t$$ (Boundary conditions, $y(\frac{\pi}{2}) = -1$, $y(0)=1$

$$s^2y(s)-sy(0)-y'(0)+9y(s)=L[\cos 2t]$$

$$(s^2+9)y(s)-s-y'(0)=\frac{s}{s^2+4}[\text{when } y(0)=1]$$

$$[L^{-1}s^2(y_s)-s(y_0)-y'(0)]+9y_{(s)}=L[\cos 2t]$$

$$(s^2+9)y_{(s)}-sy_0-y'(0)=\frac{s}{s^2+4}$$

Let $y'(0)=k$

$$(s^2+9)y_s-s-k=\frac{s}{s^2+4}$$

$$(s^2+9)y_s=\frac{s}{(s^2+4)}+s+k$$

$$y_{(s)}=\frac{s}{(s^2+4)(s^2+9)}+\frac{(s+k)}{(s^2+9)}$$

$$=\frac{1/5\,s}{(s^2+4)}-\frac{1/5\,s}{(s^2+9)}+\frac{s}{(s^2+9)}+\frac{k}{(s^2+9)}$$

$$=\frac{s}{5(s^2+4)}-\frac{s}{5(s^2+9)}+\frac{s}{(s^2+9)}+\frac{k}{(s^2+9)}$$

$$y_{(s)}=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{k}{3}\sin 3t$$

When $t=\pi/2$

$$y(\pi/2)=\frac{1}{5}(-1)-\frac{1}{5}(0)+0+\frac{k}{3}(-1)$$

$$-1+\frac{1}{5}=-\frac{k}{3}$$

$$-\frac{4}{5}=-\frac{k}{3}$$

$$\frac{12}{5}=k$$

Solution is,

$$y_{(s)}=\frac{1}{5}\cos 2t-\frac{1}{5}\cos 3t+\cos 3t+\frac{4}{5}\sin 2t$$

(4) (i) (a) Let

$$A=\begin{bmatrix}1 & 2 & p\\ 2 & 3 & -2\\ p & 1 & -p\end{bmatrix} \quad B=\begin{bmatrix}1 & 0 & -4\\ 0 & -1 & 2\\ 2 & -1 & -2\end{bmatrix}$$

When A matrix singular

$$|A|=0$$

$$|A|=\begin{vmatrix}1 & 2 & p\\ 2 & 3 & -2\\ p & 1 & -p\end{vmatrix}$$

$$\begin{aligned}&= 1(-3p+2)-2(-2p+2p)+p(2-3p)\\ &= -3p+2+2p-3p^2\\ &= -p+2-3p^2\end{aligned}$$

$$|A|=0$$

$$3p^2+p-2=0$$
$$(p+1)(3p-2)=0$$

$$p=-1 \quad p=2/3$$

(b) If $A=0$

$$A=\begin{bmatrix}1 & 2 & 0\\ 2 & 3 & -2\\ 0 & 1 & 0\end{bmatrix} \quad B=\begin{bmatrix}1 & 0 & -4\\ 0 & -1 & 2\\ 2 & -1 & -2\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 0 & -4\\ 0 & -1 & 2\\ 2 & -1 & -2\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 3 & -2\\ 0 & 1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}1 & -2 & 0\\ -2 & -1 & 2\\ 0 & -1 & 2\end{bmatrix}$$

$$BA+A=\begin{bmatrix}1 & -2 & 0\\ -2 & -1 & 2\\ 0 & -1 & 2\end{bmatrix}+\begin{bmatrix}1 & 2 & 0\\ 2 & 3 & -2\\ 0 & 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}=2I$$

$\therefore$ BA+A=2I

(c) $BA + A = 2I$

Multiplying by A^{-1}

$$BAA^{-1} + AA^{-1} = 2IA^{-1} (AA^{-1} = I)$$

$$BI + I = 2IA^{-1}$$

$$(B + I) = 2IA^{-1}$$

$$A^{-1} = \frac{1}{2I}(B + I)$$

$$A^{-1} = \frac{1}{2}(B + I)$$

$$= \frac{1}{2}\left\{\begin{bmatrix} 1 & 0 & -4 \\ 0 & -1 & 2 \\ 2 & -1 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & 2 \\ 2 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

(ii) When $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{pmatrix}$

$$L\,U = A$$

Let,

$$\begin{bmatrix} L_1 & 0 & 0 & 0 \\ L_2 & L_5 & 0 & 0 \\ L_3 & L_6 & L_8 & 0 \\ L_4 & L_7 & L_9 & L_{10} \end{bmatrix}\begin{bmatrix} U_1 & U_2 & U_3 & U_4 \\ 0 & U_5 & U_6 & U_7 \\ 0 & 0 & U_8 & U_9 \\ 0 & 0 & 0 & U_{10} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{bmatrix}$$

Here the matrix A is to be written as a product of two matrices L and U, with L lower triangular, and U upper triangular, and with one of them having all diagonal terms as 1.

That is;

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ L_2 & 1 & 0 & 0 \\ L_3 & L_6 & 1 & 0 \\ L_4 & L_7 & L_9 & 1 \end{bmatrix} \begin{bmatrix} U_1 & U_2 & U_3 & U_4 \\ 0 & U_5 & U_6 & U_7 \\ 0 & 0 & U_8 & U_9 \\ 0 & 0 & 0 & U_{10} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 2 & 4 & 0 \end{bmatrix}$$

$U_1 = 1$ $\quad L_2 . U_1 = 0$ $\quad L_2 U_3 + U_6 = 1$

$U_2 = 1$ $\quad L_2 = 0$ $\quad U_6 = 1$

$U_3 = 2$ $\quad L_2 U_2 + U_5 = 1$ $\quad L_2 U_4 + U_7 = 1$

$U_4 = 1$ $\quad U_5 = 1$ $\quad U_7 = 1$

$L_3 U_1 = 1$ $\quad L_3 U_3 + L_6 U_6 + U_8 = 1$ $\quad L_4 U_1 = 2$

$L_3 = 1$ $\quad 2 + 1 + U_8 = 1$ $\quad L_4 = 2$

$L_3 U_2 + L_6 U_5 = 2$ $\quad U_8 = -2$ $\quad L_4 U_2 + L_7 U_5 = 2$

$1 + L_6 = 2$ $\quad L_3 U_4 + L_6 U_7 + U_9 = 1$ $\quad L_7 = 0$

$L_6 = 1$ $\quad 1 + 1 + U_9 = 1$ $\quad L_4 U_3 + L_7 U_6 + L_9 U_8 = 4$

$U_9 = -1$ $\quad 4 + 0 + L_9 (-2) = 4$

$L_9 = 0$

$$L_4 U_4 + L_7 U_7 + L_9 U_0 + U_{10} = 0$$

$$2 + 0 + 0 + U_{10}$$

$$U_{10} = -2$$

Therefore,

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

$L \quad U \quad = A$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\2&0&0&1\end{bmatrix}\begin{bmatrix}1&1&2&1\\0&1&1&1\\0&0&-2&-1\\0&0&0&-2\end{bmatrix}=\begin{bmatrix}1&1&2&1\\0&1&1&1\\1&2&1&1\\2&2&4&0\end{bmatrix}$$

(5) (i) If A and B are $n\times n$ matrices,

$$\text{adj}(AB)=\text{adj B . adj A}$$

We know,

$$\text{adjB}=B^{-1}|B|$$

$$\text{adjA}=A^{-1}|A|$$

$$\begin{aligned}\text{adj B.adj A}&=B^{-1}|B|.A^{-1}|A|\\&=B^{-1}A^{-1}|B||A|\\&=(AB)^{-1}|AB| \quad \text{We know}\left[(AB)^{-1}=B^{-1}A^{-1}\right]\ \left[|AB|=|A||B|\right]\\&=adj(AB)\end{aligned}$$

$$\therefore \text{adj}(AB)=\text{adjB.adjA}$$

(ii) Find Inverse matrix, applying elementary row transformation,

$$A=\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$$

$$A = I\,A$$

$$\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A \quad R_1\to R_1-R_2\to$$

$$\begin{bmatrix}1&0&0\\2&-3&4\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\0&1&0\\0&0&1\end{bmatrix}A \quad R_2\to R_2-2R_1\to$$

$$\begin{bmatrix}1&0&0\\0&-3&4\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&0\\0&0&1\end{bmatrix}A \quad R_2\to R_2-4R_3\to$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&-4\\0&0&1\end{bmatrix}A \quad R_3 \to R_3+R_2 \to$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&-1&0\\-2&3&-4\\-2&3&-3\end{bmatrix}A$$

$$I = PA$$

According to above calculations,

$$A^{-1}=\begin{bmatrix}1&-1&0\\-2&3&-4\\-2&3&-3\end{bmatrix}$$

(iii) Without expanding

$$\begin{vmatrix}a&-b&-a&b\\b&a&-b&-a\\c&-d&c&-d\\d&c&d&c\end{vmatrix} \xrightarrow[C_2\to C_2+C_4]{C_1\to C_1+C_3} \begin{vmatrix}0&0&-a&b\\0&0&-b&-a\\2c&-2d&c&-d\\2c&2c&d&c\end{vmatrix}$$

$$\downarrow$$

$$4\begin{vmatrix}0&0&-a&b\\0&0&-b&-a\\c&-d&0&0\\d&c&0&0\end{vmatrix} \xleftarrow[C_4\to C_4-C_2]{C_3\to C_3-C_1} 4\begin{vmatrix}0&0&-a&b\\0&0&-b&-a\\c&-d&c&-d\\d&c&d&c\end{vmatrix}$$

Now, $4\{(-a)(-a)[c^2+d^2]-b(-b)[c^2+d^2]\}$

$4[c^2+d^2][a^2+b^2]$

$4(b^2+a^2)(c^2+d^2)$

(6) (i)

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} \xrightarrow[C_3 \to C_3 - C_2,\ C_4 \to C_4 - C_3]{C_2 \to C_2 - C_1} \begin{vmatrix} 1^2 & 3 & 5 & 7 \\ 2^2 & 5 & 7 & 9 \\ 3^2 & 7 & 9 & 11 \\ 4^2 & 9 & 11 & 13 \end{vmatrix}$$

$$\downarrow \quad C_4 \to C_4 - C_3,\ C_3 \to C_3 - C_2,\ C_2 \to C_2 - C_1$$

$$\begin{vmatrix} 1 & 2 & 2 & 2 \\ 4 & 1 & 2 & 2 \\ 9 & -2 & 2 & 2 \\ 16 & -7 & 2 & 2 \end{vmatrix} \xleftarrow{C_4 \to C_4 - C_3} \begin{vmatrix} 1 & 2 & 2 & 0 \\ 4 & 1 & 2 & 0 \\ 9 & -2 & 2 & 0 \\ 16 & -7 & 2 & 0 \end{vmatrix}$$

If all the elements in any row (column) of a determinant are zero then the value of the determinant is zero.

Therefore: $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} = 0$

(ii) $z = e^{xy^2}$

$\frac{\partial z}{\partial x} = e^{xy^2}.y^2$ $\qquad$ $\frac{\partial z}{\partial y} = e^{xy^2}.2yx$

$\frac{\partial z}{\partial x} = y^2 e^{xy^2}$ $\qquad$ $\frac{\partial z}{\partial y} = 2xye^{xy^2}$

$x = t\cos t$ $\qquad$ $y = t\sin t$

$\frac{dx}{dt} = t(-\sin t) + \cos t$ $\qquad$ $\frac{dy}{dt} = t\cos t + \sin t$

When $t = \pi/2$ $\qquad$ When $t = \pi/2$

$\frac{dx}{dt} = \frac{\pi}{2}(-1) + 0$ $\qquad$ $\frac{dy}{dt} = \frac{\pi}{2}(0) + 1$

$\frac{dx}{dt} = -\frac{\pi}{2}$ $\qquad$ $\frac{dy}{dt} = 1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= y^2 e^{xy^2}.\left(\frac{-\pi}{2}\right)+2xye^{xy^2}(1)$$

$$= -\frac{\pi}{2}y^2e^{xy^2}+2xye^{xy^2}$$

$$=-\frac{\pi}{2}\left[t^2\sin^2 t.e^{t\cos t.(t^2\sin^2 t)}\right]+2\left[t\cos t.t\sin t.e^{t\cos t.t^2\sin^2 t}\right]$$

$$=-\frac{\pi}{2}\left[t^2\times1\times1\times e^0\right]+2[0]\left(t=\frac{\pi}{2}\right)$$

$$=-\frac{\pi}{2}t^2$$

$$=-\frac{\pi}{2}\left(\frac{\pi}{2}\right)^2$$

$$=-\frac{\pi^3}{8}$$

(iii) $u=\dfrac{x^2y^2}{\left(x^2+y^2\right)}$

$$\frac{\partial u}{\partial x}=\frac{\left(x^2+y^2\right)2xy^2-x^2y^2.2x}{\left(x^2+y^2\right)^2}=\frac{2xy^4}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\left(x^2+y^2\right)^2 2y^4-2xy^4.2\left(x^2+y^2\right)2x}{\left(x^2+y^2\right)^4}$$

$$= \frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^4}\left[2y^4\left(x^2+y^2\right)-8x^2y^4\right]$$

$$= \frac{2y^4}{\left(x^2+y^2\right)^3}\left[y^2-3x^2\right]$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{2y^4\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\left(x^2+y^2\right)^2.2x.4y^3-2xy^4.2\left(x^2+y^2\right)2y}{\left(x^2+y^2\right)^4}$$

$$= \frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^4}\left[8xy^3\left(x^2+y^2\right)-8xy^5\right]$$

$$= \frac{8x^3y^3}{\left(x^2+y^2\right)^3}$$

$$\frac{\partial u}{\partial y}=\frac{\left(x^2+y^2\right)x^2 2y-x^2y^2 2y}{\left(x^2+y^2\right)^2}=\frac{2yx^4}{\left(x^2+y^2\right)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\left(x^2+y^2\right)^2 2x^4-2yx^4.2\left(x^2+y^2\right)2y}{\left(x^2+y^2\right)^4}$$

$$=\frac{\left(x^2+y^2\right)}{\left(x^2+y^2\right)^4}\left[2x^4\left(x^2+y^2\right)-8y^2x^4\right]=\frac{2x^4\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}$$

Prove, that,

(i) $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=x.\frac{2xy^4}{\left(x^2+y^2\right)^2}+y.\frac{2yx^4}{\left(x^2+y^2\right)^2}$$

$$=\frac{2x^2y^2\left(y^2+x^2\right)}{\left(x^2+y^2\right)^2}$$

$$=2\left[\frac{x^2y^2}{\left(x^2+y^2\right)}\right]=2u$$

(ii) $$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

$$=x^2\left[\frac{2y^4\left(y^2-3x^2\right)}{\left(x^2+y^2\right)^3}\right]+2xy\frac{8x^3y^3}{\left(x^2+y^2\right)^3}+y^2\left[\frac{2x^4\left(x^2-3y^2\right)}{\left(x^2+y^2\right)^3}\right]$$

$$=\frac{2x^2y^2}{\left(x^2+y^2\right)^3}\left[y^2\left(y^2-3x^2\right)+8x^2y^2+x^2\left(x^2-3y^2\right)\right]$$

$$=\frac{2x^2y^2}{\left(x^2+y^2\right)^3}\left[y^4+2x^2y^2+x^4\right]$$

$$=\frac{2x^2y^2}{\left(x^2+y^2\right)^3}\left(x^2+y^2\right)^2$$

$$=2\left[\frac{x^2y^2}{x^2+y^2}\right]$$

$$=2u$$

There for;

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$$

(iv) De Morgan's Theorems give,

$$\overline{X+Y} = \overline{X}\,\overline{Y}$$
$$\overline{XY} = \overline{X} + \overline{Y}$$

When taking below truth table and using AND operation and OR operation,

AND → X	Y	XY
0	0	0
0	1	0
1	0	0
1	1	1

OR → X	Y	X+Y
0	0	0
0	1	1
1	0	1
1	1	1

Can showing above details as below,

X	Y	$\overline{X}$	$\overline{Y}$	XY	$\overline{XY}$	X+Y	$\overline{X}+\overline{Y}$	$\overline{X+Y}$	$\overline{XY}$
0	0	1	1	0	1	0	1	1	1
0	1	1	0	0	0	1	1	0	1
1	0	0	1	0	0	1	1	0	1
1	1	0	0	1	0	1	0	0	0

According to above truth table, note that

$$\overline{X+Y} = \overline{X}\,\overline{Y}$$
$$\overline{XY} = \overline{X} + \overline{Y}$$