The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination – 2017/2018
Pure Mathematics - Level 04
PEU4303 - Group Theory I

Duration: - Two hours

Date: 04.04.2019 **Time: 9:30 a.m. – 11:30 a.m.**

Answer **FOUR** questions only.

1) a) State a complete set of axioms satisfied by a group.
Let $G = \mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of integers. The operation "$\star$" on G is defined as
$$(a, b) \star (c, d) = (a + c, (-1)^c b + d), \forall\ (a, b)\ , (c, d) \in G$$
Prove that (G,$\star$) is a group.
Is G an Abelian group? Justify your answer.

b) Let $G = \{(a, b): a, b \in \mathbb{C}, b \neq 0\}$
i) Let $G = \{a + ib | a, b \in \mathbb{R}$, not both zero$\}$ be a group under multiplication and let $H = \{a + ib | a^2 + b^2 = 1\}$.
Verify that H is a subgroup of G.
ii) Let $G = \{(a, b): a, b \in \mathbb{C}, b \neq 0\}$ be a group.
Show that $H = \{(na - n, a): a \neq 0, n \in \mathbb{Z}$ is a fixed positive integer$\}$ forms a subgroup of G.

2) a) Let $G = \{(a, b): a, b \in C, b \neq 0\}$. Show that G is a non abelian group under the operation "$\star$" defined by
$$(a, b) \star (c, d) = (a + bc, bd), \forall\ (a, b)\ , (c, d) \in G$$

b) Prove or disprove the following statements.
i) If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is a subgroup of G.
ii) If H_1 and H_2 are two subgroups of a group G, then it is not necessarily true that $H_1 \cup H_2$ is a subgroup of G.
iii) A subgroup of a cyclic group is cyclic.

3) a) Define the following terms
i) The order of a group
ii) The order of an element
b) Let G= {1,2,3,4,5,6} be a group under the operation $\otimes_7$defined by $a \otimes_7 b = r, 0 < r < 7$; where r is the remainder when ordinary multiplication ab is divided by 7.
i) By using composition table show that $(G, \otimes_7)$ is a group.

ii) Find the order of each element in G.
iii) Show that $(G, \otimes_7)$ is cyclic.
iv) How many generators are there of the cyclic group $(G, \otimes_7)$? Justify your answer.

4) State Lagrange's Theorem.
a) Find the order of each element in the group $(\mathbb{Z}_6, \oplus_6)$.
Does $\mathbb{Z}_6$ form a cyclic group? Justify your answer.

b) Showing steps clearly, prove the following statements.
i) Every cyclic group is necessarily abelian.
ii) If a is a generator of a cyclic group G, then a^{-1} is also a generator of G.
iii) The order of a finite cyclic group is equal to the order of any generator of the group.

5) a) Prove each of the following statements.
If $(G, \star)$ is a group, then
i) the identity element of G is unique.
ii) every $a \in G$ has a unique inverse in G.
iii) for every $a \in G$, $(a^{-1})^{-1} = a$.
iv) for all $a, b \in G$, $(a \star b)^{-1} = b^{-1} \star a^{-1}$.
v) for all $a, b, c \in G$, $ab = ac \Rightarrow b = c$ and $ba = ca \Rightarrow b = c$.

b) Determine the subgroup lattice for $\mathbb{Z}_{p^2q}$, where p and q are distinct primes.
c) Find all generators of $\mathbb{Z}_6, \mathbb{Z}_8$ and $\mathbb{Z}_{20}$.

6) a) Let G be a group and $Z(G) = \{x \in G | xg = gx \text{ for all } g \in G\}$ then $Z(G)$ be the center of the group G.
Show that $Z(G)$ is a subgroup of G.

b) i) Show that $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | \, a, b, c, d \in \mathbb{R}, \; ad - bc \neq 0 \right\}$ is a group under the matrix multiplication.

ii) If $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in G, ad \neq 0 \right\}$ then show that H is a subgroup of G.

c) Let G be a group. If $a^5 = e$ and $aba^{-1} = b^2$ for $a, b \in G$, then show that order of b is 31.