The Open University of Sri Lanka
Department of Mathematics
B. Sc/ B. Ed Degree Programme
Final Examination - 2017/ 2018
Pure Mathematics– Level 05

PEU5300/PUU3143– Riemann Integration

Duration: Two Hours

Date: 07 – 04 – 2019 **Time: 01.30 p.m. – 03.30 p.m.**

ANSWER FOUR QUESTIONS ONLY

Q1)

(a) Let f be a bounded function on $[a, b]$. Prove that f is Riemann integrable on $[a, b]$ if and only if for each $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

(b) Let f and g be integrable functions on $[a, b]$. Prove that

(i) $f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$,

(ii) $-f$ is integrable and $\int_a^b (-f) = -\int_a^b f$.

(c) Is the function f defined by $f(x) = \begin{cases} -1, & x \in [0,1] \cap \mathbb{Q} \\ 1, & x \in [0,1] \cap \mathbb{Q}^c \end{cases}$, Riemann integrable? Justify your answer.

Q2)

(a) Let $f: [1, 2] \to \mathbb{R}$ be the function defined by $f(x) = x$.

(i) Find the upper and lower Riemann integrals of f on $[1, 2]$.

(ii) Show that f is Riemann integrable on $[1, 2]$ and $\int_1^2 f(x)dx = {}^3/_2$.

(b) Let $f: [a, b] \to \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 2 & , x \in [1,3] \cap \mathbb{Q} \\ x + 1, & x \in [1,3] \cap \mathbb{Q}^c \end{cases}$

(i) Find the upper and lower Riemann integrals of f on $[1, 3]$.

(ii) Is f Riemann integrable on $[1, 3]$? Justify your answer.

Q3)

(a) Let $f: [a, b] \to \mathbb{R}$ be a function such that $|f(x)| \leq M$ for all $x \in [a, b]$.

(i) Show that $U(P, f^2) - L(P, f^2) \leq 2M[U(P, f) - L(P, f)]$ for all partitions P of $[a, b]$.

(ii) Show that if f is integrable on $[a, b]$ then f^2 is also integrable on $[a, b]$.

(b) Using above result or otherwise show that if g and h are integrable functions on $[a,b]$ then gh is also integrable on $[a,b]$.

(c) Suppose f is a bounded real function on $[a,b]$, and f^2 is integrable on $[a,b]$. Does it follow that f is integrable? Justify your answer.

Q4)

(a) State and prove the Mean Value Theorem for integrable functions.

Suppose that f and g are continuous functions on $[a,b]$ such that $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$. Prove that there exists $c \in (a,b)$ such that $f(c) = g(c)$.

(b) Let $f: [a,b] \to \mathbb{R}$ be a bounded function.

Show that $U(P,|f|) - L(P,|f|) \leq [U(P,f) - L(P,f)]$ for all partitions P of $[a,b]$.

Hence or otherwise show that if f is integrable on $[a,b]$ then $|f|$ is also integrable on $[a,b]$ and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Q5)

(a) Suppose that

$$f(t) = \begin{cases} t, & t < 0 \\ t^2 + 1, & 0 \leq t \leq 2 \\ 0, & t > 2 \end{cases}$$

(i) Find a formula for $F(x)$ in $[0,6]$, where $F(x) = \int_0^x f(t)\,dt$.

(ii) Where is F differentiable? Justify your answer.

(b) Let f be a Riemann integrable function on $[a,b]$. Prove that the function $F: [a,b] \to \mathbb{R}$, defined by $F(x) = \int_a^x f(t)\,dt$ is uniformly continuous on $[a,b]$.

(c) Let f be a continuous function on $\mathbb{R}$ and let F be defined by $F(x) = \int_{x-2}^{x+2} f(t)\,dt$ for $x \in \mathbb{R}$. Show that F is differentiable on $\mathbb{R}$ and find F'.

Q6) Consider the following integrals:

(i) $\int_0^1 \ln x\,dx$, (ii) $\int_{-1}^{7} \frac{1}{\sqrt[3]{x+1}}dx$, (iii) $\int_{-\infty}^{+\infty} \frac{1}{x^2+1}dx$, (iv) $\int_{-\infty}^{-4} \frac{1}{x+3}dx$.

(a) Explain why each of the above is an improper integral.

(b) Determine the convergence of them and find the respective values when converge.

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