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The Open University of Sri Lanka
B.Sc/B.Ed. Degree Programme
Final Examination - 2017/2018
Pure Mathematics - Level 05
PEU5303-Number Theory

Duration: - Two hours

Date: 31-03-2019 **Time: 1:30 p.m. -3:30 p.m.**

Answer Four Questions Only

1. (a) (i) State the Well Ordering Principle.

(ii) Let $A = \{a + b : \ a, b$ are positive integers such that, $a^2 + b^2 = 10001\}$ and $B = \{b: \ b$ is a positive integer such that $(b^2 + 1)$ is a multiple of 3 $\}$.
Prove that the set A has a least element but the set B does not have a least element.

(b) Use Euclidean Algorithm to find gcd(987, 144). Find $x, \ y \in \mathbb{Z}$ such that $\gcd(987, \ 144) = 987x + 144y$.

(c) A house wife spends 1770 Rupees for purchasing mangoes and apples. She pays 31 Rupees for each mango and 21 Rupees for each apple. If she buys more mangoes than apples, how many mangoes and apples she buys?

2. (a) Find all positive integers m that satisfy the congruence $113 \equiv 8 \pmod{m}$.

(b) State the Division Algorithm.

(c) (i) If m and n are odd integers then prove that $8|(m^2 - n^2)$.

(ii) If p , q are not divisible by 3 then prove that $24|(p^2 - q^2)$.

(d) If p is a prime number greater than 3, then prove that p is of the form $6k + 1$ or $6k + 5$, where k is an integer.

3. (a) State the necessary and sufficient condition for existing solutions to the linear congruence $ax \equiv b \pmod{n}$

(b) Find all incongruent solutions of the following linear congruences .

(i) $18x \equiv 30 \pmod{42}$

(ii) $7x \equiv 5 \pmod{11}$

(iii) $9x \equiv 12 \pmod{15}$

(c) Use Chinese remainder theorem to solve the following problem;
One egg remains when the eggs are removed from an egg basket 2, 3 or 5 at a time. But no eggs remains if they are removed 7 at a time. Find the smallest number of eggs that could have been in the basket.

4. (a) (i) Prove that there are infinitely many prime numbers.

(ii) Prove that for each $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ such that $m + n$ is not prime.

(iii) Prove that for each $m \in \mathbb{N}$, there exist $n \in \mathbb{N}$ such that $m + n$ is prime.

(b) (i) Prove that $\sqrt{2}$ is irrational.

(ii) Prove that between any two irrational numbers there is an irrational number.

5. (a) By showing your work, find each of the following.

(i) $2^{83} \pmod{17}$

(ii) $3^{29} \pmod{31}$

(iii) $2^{341} \pmod{11}$

(iv) $19^{19} \pmod{17}$

{Note that $a \pmod{m}$ means the remainder after dividing a by m}

(b) Find a complete residue system modulo 11 which consist of

(i) Odd numbers only

(ii) Even numbers only

6. (a) (i) State the Fermat's Little Theorem

(ii) State the Wilson's Theorem

(b) Prove that $a^{13} - a$ is divisible by 2 , 3, 5, 7 and 13 for any integer a.

(c) Let a be any integer with $\gcd(a, 561) = 1$. Prove each of the followings.

(i) $a^{560} \equiv 1 \pmod 3$

(ii) $a^{560} \equiv 1 \pmod{11}$

(iii) $a^{560} \equiv 1 \pmod{17}$

Hence, deduce that $a^{560} \equiv 1 \pmod{561}$

(d) If p is a prime number then find $(p-2)! \pmod p$